LARGE TIME ASYMPTOTICS OF SOLUTIONS TO THE CAUCHY PROBLEM FOR THE FRACTIONAL MODIFIED KDV EQUATION

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We study the fractional modified Korteweg-de Vries equation

$$\left\{ \begin{array}{c} \partial_t u + \frac{1}{\alpha} \left|\partial_x\right|^{\alpha-1} \partial_x u = \partial_x \left(u^3\right), \ t > 0, \ x \in \mathbb{R}, \\ u\left(0, x\right) = u_0\left(x\right), \ x \in \mathbb{R}, \end{array} \right. \tag{1}$$

where $\alpha \in (0,1)$, $\left|\partial_x\right|^{\alpha} = \mathcal{F}^{-1} \left|\xi\right|^{\alpha} \mathcal{F}$ is the fractional derivative. The case of $\alpha = 3$ corresponds to the classical modified KdV equation. In the case of $\alpha = 2$, (1) is called the modified Benjamin-Ono equation. Fractional KdV equation was proposed in [1] as a model approximation for the Whitham equation. It is known that the Whitham equation describes the full dispersion

$$\xi \sqrt{\frac{\tanh \xi}{\xi}}$$

of water waves. Then in the short wave limit $|\xi| \to \infty$,

$$\xi \sqrt{\frac{\tanh \xi}{\xi}} = \xi \sqrt{\frac{\left(e^{\xi} - e^{-\xi}\right)}{\xi \left(e^{\xi} + e^{-\xi}\right)}} \to |\xi|^{-\frac{1}{2}} \xi$$

we arrive at (1) with $\alpha = \frac{1}{2}$.

The fractional modified KdV equation (1) has been considered in [1] (J.-C. Saut and Y. Wang, *Long time behavior of the fractional Korteweg–de Vries equation with cubic nonlinearity.* Discrete Contin. Dyn. Syst. **41** (2021), no. 3, 1133–1155).

Define the free evolution group associated with (1) by $\mathcal{U}(t) = \mathcal{F}^{-1} e^{-it\Lambda(\xi)} \mathcal{F}$, where the symbol $\Lambda(\xi) = \frac{1}{\alpha} |\xi|^{\alpha-1} \xi$. They showed

Proposition 1. *Let $\alpha \in (0,1)$ and define the Z - norm*

$$\|g\|_Z = \left\| \left(1 + |\xi|\right)^{10} \left(\mathcal{F} g\right)(\xi) \right\|_{\mathbf{L}^{\infty}_{\xi}}.$$

Assume that $N_0 = 100$, $p_0 \in \left(0, \frac{1}{1000}\right] \cap \left(0, \frac{1-\alpha}{100}\right]$ are fixed, and $u_0 \in \mathbf{H}^{N_0}$ is real and satisfies

$$\|u_0\|_{\mathbf{H}^{N_0}} + \|u_0\|_{\mathbf{H}^{1,1}} + \|u_0\|_Z = \varepsilon_0 \leq \overline{\varepsilon}$$

for some constant $\overline{\varepsilon}$ sufficiently small (depending only on α and p_0). Then the Cauchy problem (1) admits a unique global solution $u \in \mathbf{C}\left(\mathbb{R}; \mathbf{H}^{N_0}\right)$ satisfying the following uniform bounds for $t \geq 1$

$$t^{-p_0} \|u(t)\|_{\mathbf{H}^{N_0}} + t^{-p_0} \|\varphi\|_{\mathbf{H}^{1,1}} + \|\varphi\|_Z \leq C\varepsilon_0,$$

Key words and phrases. Fractional modified KdV equation, Modified scattering, Asymptotics for large time.

35B40; 35Q92, This a a joint work with Naumkin.

where $\varphi(t) = \mathcal{U}(-t)\, u(t)$. *Moreover there exists* $w_\infty \in \mathbf{L}^\infty$ *such that for* $t \geq 1$

$$(2)\quad t^{p_0} \left\| \exp\left(-\frac{i\xi\, |\xi|^{2-\alpha}}{1-\alpha} \int_1^t |\widehat{\varphi}(\tau,\xi)|^2 \frac{d\tau}{\tau} \right) (1+|\xi|)^{10}\, \widehat{\varphi}(t,\xi) - w_\infty(\xi) \right\|_{\mathbf{L}^\infty_\xi} \leq C\varepsilon.$$

From the estimate presented in Proposition 1 and Lemma 2.2 in [1], $\mathbf{L}^\infty$ time decay of solutions such that $\|u(t)\|_{\mathbf{L}^\infty} \leq C\varepsilon_0 \langle t\rangle^{-\frac{1}{2}}$ follows. However, we note that formula (2) does not say the large time asymptotics of solutions to (1). Our aim is to relax the regularity conditions on the initial data and to find the large time asymptotic formula for solutions to (1) in an explicit form with the estimates of the remainder terms.

Our functional space for solutions is based on the following norm

$$\|u\|_{\mathbf{X}_T} = \sup_{t\in[0,T]} \left(\left\| \langle\xi\rangle^2 \widehat{\varphi}(t) \right\|_{\mathbf{L}^\infty} + \langle t\rangle^{-\varepsilon} \|\mathcal{J}u(t)\|_{\mathbf{H}^3} + \langle t\rangle^{-\varepsilon} \|u(t)\|_{\mathbf{H}^8} \right),$$

where $\varepsilon > 0$ is small which depends on the size of the data of the norm of $\mathbf{X}_0$, $\widehat{\varphi}(t) = \mathcal{F}\mathcal{U}(-t)\, u(t)$ and the operator $\mathcal{J} = \mathcal{U}(t)\, x\, \mathcal{U}(-t)$.

The weighted Sobolev space is

$$\mathbf{H}_p^{m,s} = \left\{ \varphi \in \mathbf{S}'; \|\phi\|_{\mathbf{H}_p^{m,s}} = \|\langle x\rangle^s \langle i\partial_x\rangle^m \phi\|_{\mathbf{L}^p} < \infty \right\},$$

$m, s \geq 0, 1 \leq p \leq \infty$, $\langle x\rangle = \sqrt{1+x^2}, \langle i\partial_x\rangle = \sqrt{1-\partial_x^2}$. We also use the notations $\mathbf{H}^{m,s} = \mathbf{H}_2^{m,s}$, $\mathbf{H}^m = \mathbf{H}^{m,0}$. Let $\mathbf{C}(\mathbf{I};\mathbf{B})$ be the space of continuous functions from the time interval $\mathbf{I}$ to a Banach space $\mathbf{B}$.

To state our main result we introduce the dilation operator $\mathcal{D}_t\phi(x) = t^{-\frac{1}{2}}\phi\left(\frac{x}{t}\right)$, the scaling operator $(\mathcal{B}\phi)(x) = \phi(\mu(x))$, where the stationary point $\mu(x) = x\,|x|^{-\frac{2-\alpha}{1-\alpha}}$, and the multiplication factor $M = e^{it\left(1-\frac{1}{\alpha}\right)|\eta|^\alpha}$. Define the Heaviside function $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x \leq 0$. Denote $\Theta(\xi) = \frac{3\xi}{|\Lambda''(\xi)|}$.

The main result of our paper is the following.

Theorem 1. *Let the initial data* $u_0 \in \mathbf{H}^8 \cap \mathbf{H}^{3,1}$ *be real valued and* $0 < \|u_0\|_{\mathbf{H}^8\cap\mathbf{H}^{3,1}} < \varepsilon$. *Then there exists an* ε *such that (1) has a unique global solution* u *such that* $u \in \mathbf{C}\left([0,\infty);\mathbf{H}^8 \cap \mathbf{H}^{3,1}\right)$. *Moreover for any* u_0, *there exists a unique modified final state* $W_+ \in \mathbf{L}^\infty$ *such that the asymptotics*

$$(3)\qquad u(t) = 2Re\mathcal{D}_t\mathcal{B}M\theta\sqrt{\frac{i}{|\Lambda''|}}W_+ \exp\left(i\frac{3\xi}{|\Lambda''(\xi)|} |W_+|^2 \log t \right) + O\left(\varepsilon t^{-\frac{1}{2}-\delta}\right)$$

is valid for $t \to \infty$ *uniformly with respect to* $x \in \mathbb{R}$, *where* $0 < \delta < \frac{1}{4}$.

Remark 1. *Theorem 1 says the main term of solutions vanishes in the negative line and the estimate of solutions in the negative line is included in the remainder term. This fact comes from* $\mathcal{U}(t)\,\phi$ *decays rapidly in the negative line.*

References

[1] J.-C. Saut and Y. Wang, *Long time behavior of the fractional Korteweg–de Vries equation with cubic nonlinearity.* Discrete Contin. Dyn. Syst. **41** (2021), no. 3, 1133–1155.

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