Weighted Trudinger-Moser inequalities in the subcritical Sobolev spaces and their applications

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Let $1 < p \le N, p' := \frac{p}{p-1}$, B_R^N be open ball in $\mathbb{R}^N$ with center 0, with radius $R \in (0, \infty)$ and ω_N be the surface area of unit sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$. For unified notation, we set $B_\infty^N = \mathbb{R}^N$. The boundedness, optimality and the existence of a maximizer of the following type maximization problems $T_p, T_p^{\rm rad}$ have been studied so far.

$$T_p := \sup \left\{ \int_{B_R^N} f(u) V(|x|)\, dx \,\middle|\, u \in \dot{W}_0^{1,p}(B_R^N),\ \|\nabla u\|_p \le 1 \right\}$$

$$\ge \sup \left\{ \int_{B_R^N} f(u) V(|x|)\, dx \,\middle|\, u \in \dot{W}_{0,\rm rad}^{1,p}(B_R^N),\ \|\nabla u\|_p \le 1 \right\} =: T_p^{\rm rad}$$

First, we show the following figures to explain known results about the boundedness and our motivations.

Table 1: Boundedness of $T_p^{\rm rad}$ $(p < N, R \in (0, \infty])$

$f(u)$	$V(\|x\|)$	Name of ineq.	Supplement
$\|u\|^p$	$\|x\|^{-p}$	Hardy	
$\|u\|^q \ (p < q < p^*)$	$\|x\|^{-A_q}$	Hardy-Sobolev	$A_q := \frac{p^*-q}{p^*-p} p > 0$
$\|u\|^{p^*}$	1	Sobolev	$p^* := \frac{Np}{N-p}$
$\|u\|^q \ (p^* < q < \infty)$	$\|x\|^{B_q}$	(Hénon or Ni)	$B_q := \frac{q-p^*}{p^*-p} p > 0$

Table 2: Boundedness of T_N and $T_N^{\rm rad}$ $(R \in (0, \infty))$

$f(u)$	$V(\|x\|)$	Name of ineq.	Supplement
$\|u\|^N$	$\|x\|^{-N} \left(\log \frac{aR}{\|x\|}\right)^{-N}$	Critical Hardy	$a \ge 1$
$\|u\|^q \ (N < q < \infty)$	$\|x\|^{-N} \left(\log \frac{aR}{\|x\|}\right)^{-\beta_q}$	(Generalized C.H.)	$\beta_q := \frac{N-1}{N} q + 1$
$\exp(\gamma_\beta \|u\|^{N'})$	$\|x\|^{-\beta} \ (\beta \in (0, N))$	Singular T.-M.	$\gamma_\beta := \alpha_N (1 - \beta/N)$
$\exp(\alpha_N \|u\|^{N'})$	1	Trudinger-Moser	$\alpha_N := N \omega_N^{\frac{1}{N-1}}$

In this talk, we consider the boundedness, optimality and the attainability of $T_p^{\rm rad}$ with the exponential growth $f(u) = \exp(\alpha |u|^{p'})$ beyond the polynomial growth $|u|^q$ in the subcritical case $p < N$, see Table 1. This is an analogue of the critical case $p = N$, see Table 2. More precisely, we give a suitable weight function $V_p(|x|)$ which is more rapidly vanishing at 0 than the Hénon weight function $|x|^{B_q}$. Furthermore, our inequality converges to the original Trudinger-Moser inequality as $p \nearrow N$ including the optimal exponent α_N and the concentration limit $1 + e^{1+\frac{1}{2}+\cdots\frac{1}{N-1}}$. If time permits, we also give applications of our inequality to the elliptic and the parabolic equations with exponential nonlinearity.

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