ABSTRACT: STABILITY OF INFINITE QUANTUM SYSTEMS

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This work is based on the joint work with Antoine Borie and Julien Sabin (Univ. of Rennes).

1. Introduction

1.1. Equation and its background. In this talk, we consider the following Hartree equation:

$$i\partial_t\gamma = [-\Delta + w * \rho_\gamma, \gamma], \quad \gamma : \mathbb{R} \to \mathcal{B}(L^2_x), \tag{NLH}$$

where $\mathcal{B}(L^2_x)$ is the space of all bounded operators on $L^2_x = L^2_x(\mathbb{R}^d)$, w is a given finite signed measure on $\mathbb{R}^d$, and $\rho_\gamma(t,x) := \gamma(t,x,x)$ ($\gamma(t,x,y)$ is the integral kernel of $\gamma(t)$).

The time evolution of the N fermions in $\mathbb{R}^d$ is described by the linear Schrodinger equation in $\mathbb{R}^{Nd}$. It is too complicated, but we can approximate it by the Hartree (or, more precisely, reduced Hartree–Fock equation):

$$i\partial_t u_n = (-\Delta_x + w * \rho)u_n, \quad \rho = \sum_{j=1}^{N} |u_j|^2, \quad u_n : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}, \quad (n = 1, \dots N). \tag{1}$$

Let $(u_n(t))_{n=1}^N \subset C(\mathbb{R}; L^2_x)$ be a solution to (1). Setting $\gamma = \sum_{n=1}^N |u_n(t)\rangle\langle u_n(t)|$, we can easily derive (NLH).

1.2. Existence of infinitely many stationary solutions. The following fact is pretty important:

Proposition 1. Let $d \geq 1$. If w is a finite signed measure on $\mathbb{R}^d$ and $f \in L^1_\xi \cap L^\infty_\xi$, then $\gamma_f := \mathcal{F}^{-1} f \mathcal{F}$ is a stationary solution to (NLH).

By Proposition (1), we have infinitely many stationary solutions to (NLH). Moreover, if $f \geq 0$ and $f \not\equiv 0$, we can interpret each γ_f is an *infinite quantum system* because

$$\mathrm{Tr}(\gamma_f) = \text{“the number of particles”} = \infty.$$

There are some physically important examples of f. For example, $f(\xi) = \mathbb{1}_{\{|\xi|^2 \leq \mu\}}$ for Fermi gas at 0[K], or $f(\xi) = \frac{1}{e^{(|\xi|^2-\mu)/T}+1}$ for Fermi gas at T[K], where μ is chemical potential.

1.3. Aim of this talk and known results. In this talk, we prove asymptotic stability of γ_f for a wide class of f and w. This is an quantum analogy of the nonlinear Landau damping for the Vlasov equation.

In [2014, Lewin–Sabin (APDE)], this problem was *first* formulated and given a significant result when $d = 2$. This result was extended to $d \geq 3$ in [2018, Chen–Hong–Pavlović (AIHP)]. However, the above results required smoothness of w and f; hence, the physically important example $f = \mathbb{1}_{\{|\xi|^2 \leq 1\}}$ is excluded. The speaker weakened the assumptions for w and f in [2023, H. Preprint], and included $f = \mathbb{1}_{\{|\xi|^2 \leq 1\}}$. We can consider the similar equation in the random fields setting ([2020, Collot-de Suzzoni, JMPA], [2022, Collot-de Suzzoni (Ann. Henri Lebesgue)]).

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2. Main result

2.1. Preliminaries.

Definition 1. If $T : L^2_x \to L^2_x$ is compact, we can write $Tu = \sum_{n=1}^\infty a_n \langle f_n | u \rangle g_n$, where $(f_n)_{n=1}^\infty$, $(g_n)_{n=1}^\infty$ are O.N.S. in L^2_x, and $a_n \geq a_{n+1} \downarrow 0$. We define

$$\|T\|_{\mathfrak{S}^\alpha} := \|a_n\|_{\ell^\alpha_n} \quad \text{for any } \alpha \in [1, \infty].$$

Moreover, we write $\|Q_0\|_{\mathcal{H}^{s,\alpha}} := \|\langle \nabla \rangle^s Q_0 \langle \nabla \rangle^s\|_{\mathfrak{S}^\alpha}$, $\mathcal{H}^s := \mathcal{H}^{s,2}$.

Definition 2. For f and w, we define $L = L(f, w)$ by

$$L[g](t) = -i\rho \left(\int_0^t U(t - t_1)_\star [w * g(t_1), \gamma_f] dt_1 \right),$$

where $A_\star B := ABA^*$ and $U(t) := e^{it\Delta}$.

2.2. Statement.

Let $\gamma = Q + \gamma_f$. Then, we have

$$i\partial_t Q = [-\Delta + w * \rho_Q, Q + \gamma_f]. \qquad (f\text{-NLH})$$

Theorem 1 (2024, Borie–H.–Sabin, In preparation)**.** *Let $d \geq 3$ and $s = \frac{d}{2} - 1$. Let w be a finite signed measure on $\mathbb{R}^d$, and $\|\langle \xi \rangle^{4s} f(\xi)\|_{L^1_\xi \cap L^\infty_\xi} < \infty$. Assume that $(1 - L(f, w))^{-1} \in L^2_t(\mathbb{R}_+; L^2_x)$. Then, there exists $\varepsilon_0 > 0$ such that the following holds: If $Q_0 \in \mathcal{H}^s$ satisfies $\|\rho(U(t)_\star Q_0)\|_{L^2_t H^s_x} \leq \varepsilon_0$, there exists a unique solution $Q(t) \in C(\mathbb{R}_+; \mathcal{H}^s)$ to $(f$-NLH$)$ such that $\|\rho_Q\|_{L^2_t(\mathbb{R}_+; H^s_x)} \leq 2\varepsilon_0$. Moreover, $Q(t)$ sucatters, that is, there exists $Q_+ \in \mathfrak{S}^{\frac{2d}{d-1}}$ such that*

$$U(t)^* Q(t) U(t) \to Q_+ \text{ in } \mathfrak{S}^{\frac{2d}{d-1}} \text{ as } t \to \infty.$$

2.3. Comments on $(1 - L)^{-1}$.

We have $(1 - L(f, w))^{-1} \in \mathcal{B}(L^2_t(\mathbb{R}; L^2_x))$ if the quantum Penrose stability condition (QPSC) is satisfied:

$$\inf_{(s,\tau,\xi) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^d} \left| 1 + 2\widehat{w}(\xi) \int_0^\infty e^{-(s+i\tau)t} \sin(t|\xi|^2) \check{f}(2t\xi) dt \right| > 0.$$

A sufficient condition of (QPSC) is

$$\|\widehat{w}\|_{L^\infty_\xi} \int_{\mathbb{R}^d} \frac{|\check{f}(\xi)|}{|\xi|^{d-2}} d\xi \ll 1.$$

When $d = 3$ and $f(\xi) = \mathbb{1}_{\{|\xi|^2 \leq 1\}}$ (Fermi gas at zero temperature), we have

$$\int_{\mathbb{R}^d} \frac{|\check{f}(\xi)|}{|\xi|^{d-2}} d\xi = \infty.$$

However, we have

Proposition 2 (2023, H. Preprint and 2024, Borie–H.–Sabin, In preparation)**.** Let $d = 3$ and $f(\xi) = \mathbb{1}_{\{|\xi|^2 \leq 1\}}$. Then, $-1 \ll \widehat{w} \leq 0$ implies (QPSC).