A PDE-based approach to Borell-Brascamp-Lieb inequality

Qing Liu

Okinawa Institute of Science and Technology

In this talk, we provide a new PDE perspective for the celebrated Borell-Brascamp-Lieb inequality, which is stated as follows.

Theorem 1. *Let $n \geq 1$, $\lambda \in (0,1)$, $\alpha \in [-1/n, \infty]$. Assume that f, $g, h \in L^1(\mathbb{R}^n)$ are nonnegative and satisfy*

$$h((1-\lambda)y + \lambda z) \geq M_\alpha(f(y), g(z); \lambda), \qquad \forall y,\ z \in \mathbb{R}^n.$$

Then,

$$\int_{\mathbb{R}^n} h(x)\,dx \geq M_{\frac{\alpha}{n\alpha+1}}\left(\int_{\mathbb{R}^n} f(x)\,dx, \int_{\mathbb{R}^n} g(x)\,dx; \lambda\right).$$

In this theorem, $M_\alpha(a, b; \lambda)$ denotes the power mean associated to given $\alpha \in [-\infty, +\infty]$, $a, b \geq 0$ and $\lambda \in (0,1)$, that is, we set $M_\alpha(a, b; \lambda) = 0$ if $ab = 0$, and

$$M_\alpha(a, b; \lambda) = \begin{cases} \max\{a, b\} & \alpha = +\infty \\ ((1-\lambda)a^\alpha + \lambda b^\alpha)^{\frac{1}{\alpha}} & \alpha \neq 0, \pm\infty \\ a^{1-\lambda} b^\lambda & \alpha = 0 \\ \min\{a, b\} & \alpha = -\infty \end{cases}$$

if $ab > 0$.

This fundamental functional inequality in its general form was studied by [2, 3]. The special case with $\alpha = 0$ is known as the Prékopa-Leindler inequality, which is closely related to the Brunn-Minkowski inequality and has significant applications in convex geometry. See [4] for an excellent survey concerning related topics on these inequalities.

Previously known proofs of Theorem 1 involve techniques from convex analysis or optimal transport. In contrast, our new proof is based on properties of diffusion equations of porous medium type

$$\partial_t u - \frac{1}{m}\Delta(u^m) = 0,$$

where $m \in \mathbb{R}$ is chosen as

$$m = 2\alpha + 1$$

in accordance with the exponent α appearing in the Borell-Brascamp-Lieb inequality. We focus on the case of $\alpha < 0$, which consequently requires m to lie in the fast diffusion regime $m < 1$. The main tools we use in our proof are a generalized concavity maximum principle and large time asymptotics for the equation.

Our approach reveals a deep connection between the Borell-Brascamp-Lieb inequality and nonlinear parabolic equations. An analogous idea was introduced in [1] to prove the special case of the Prékopa-Leindler inequality by using the heat equation. Our analysis serves as a nonlinear generalization and improvement of this PDE approach.

Furthermore, our work also demonstrates the possiblity of using PDEs to understand equality conditions for functional inequalities. While it is not completely clear to us how to employ fast diffusions to retrieve the equality condition for the general Borell-Brascamp-Lieb inequality, we manage to recover the equality condition for the Prékopa-Leindler inequality by further exploiting additional properties of the heat equation including the eventual log-concavity and backward uniqueness of solutions.

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References

[1] F. Barthe, D. Cordero-Erausquin, Inverse Brascamp-Lieb inequalities along the heat equation, Geometric aspects of functional analysis, *Lecture Notes in Math.*, 1850 (2004), 65–71.

[2] C. Borell, Convex set functions in d-space, *Period. Math. Hungar.* 6 (1975), 111–136.

[3] H. J. Brascamp, E. H. Lieb, On extensions of the Brunn-Minkowski and Prékopa-Leindler theo- rems, including inequalities for log concave functions, and with an application to the diffusion equation, *J. Functional Analysis* 22 (1976), 366–389.

[4] R. J. Gardner. The Brunn-Minkowski inequality, *Bull. Amer. Math. Soc. (N.S.)*, 39 (2002), 355–405.