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Corrections to “Twisted cohomology of a punctured Riemann surface”

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Correction: Kumamoto J. Math. 29 (2016):55-63

In a previous paper [1], the following corrections of the text should be made.

p. 56, l. 2 from the top: “where we give” *should read* “which includes information on how to give”.
p. 61, l. 10 from the top: “$H^1(X^*, \mathcal{L})$” *should read* “$\mathrm{gr}\, H^1(X^*, \mathcal{L})$”.
p. 61, l. 9 from the bottom: “Therefore” *should read* “Then we see that”.
p. 61, l. 8 from the bottom: “$H^1(X^*, \mathcal{L})$.” *should read* “$H^1(X^*, \mathcal{L})$ as well as $\mathrm{gr}\, H^1(X^*, \mathcal{L})$.”.
p. 62, l. 3 from the top: “Therefore” *should read* “Then we see that”.
p. 62, l. 4 from the top: “$H^1(X^*, \mathcal{L})$.” *should read* “$H^1(X^*, \mathcal{L})$ as well as $\mathrm{gr}\, H^1(X^*, \mathcal{L})$.”.
p. 62, l. 11 from the bottom: “$H^1(X^*, \mathcal{L})$” *should read* “$\mathrm{gr}\, H^1(X^*, \mathcal{L})$”.
p. 62, l. 5 from the bottom: “of order 2.” *should read* “of order 2 with residue zero.”.

These changes correct the statements of Theorem 4.1 and Corollary 5.1 in [1], and do not affect the other parts of the text.

And we can derive more information on the structure of the group $H^1(X^*, \mathcal{L})$ from the corrected Theorem 4.1.

First, we consider the case where $P \neq 1$. Since $\nabla H^0(X, \mathcal{O}_X(D)(P)) \cap H^0(X, \Omega_X^1(D)(P)) = 0$, we have by the corrected Theorem 4.1 (i) a short exact sequence

$$0 \longrightarrow E_\infty^{10} \longrightarrow \frac{H^0(X, \Omega_X^1(2D)(P))}{\nabla H^0(X, \mathcal{O}_X(D)(P))} \longrightarrow E_\infty^{01} \longrightarrow 0. \tag{6}$$

Next, we consider the case where $P = 1$. Since $\nabla H^0(X, \mathcal{O}_X(D)) \cap H^0(X, \Omega_X^1(D)) = \boldsymbol{C}\omega$, we have by the corrected Theorem 4.1 (ii) a short exact sequence

$$0 \longrightarrow E_\infty^{10} \longrightarrow \frac{H^0(X, \Omega_X^1(2D))}{\nabla H^0(X, \mathcal{O}_X(D))} \longrightarrow E_\infty^{01} \longrightarrow 0. \tag{7}$$

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Comparing (6) and (7) with the short exact sequence

$$0 \longrightarrow E_\infty^{10} \longrightarrow H^1(X^*, \mathcal{L}) \longrightarrow E_\infty^{01} \longrightarrow 0,$$

we arrive at the following

Theorem 4.2. In either case $P = 1$ or $P \neq 1$, we have a canonical isomorphism

$$H^1(X^*, \mathcal{L}) \cong \frac{H^0(X, \Omega_X^1(2D)(P))}{\nabla H^0(X, \mathcal{O}_X(D)(P))}.$$

Proof. According to Chap. XIV in [2], it suffices to prove the existence of a natural homomorphism

$$\frac{H^0(X, \Omega_X^1(2D)(P))}{\nabla H^0(X, \mathcal{O}_X(D)(P))} \longrightarrow H^1(X^*, \mathcal{L}).$$

In fact, it follows from the standard theory of de Rham that

$$H^1(X^*, \mathcal{L}) \cong \frac{\operatorname{Ker}[\nabla : \Gamma(X^*, \mathcal{E}_{X^*}^1(P|X^*)) \longrightarrow \Gamma(X^*, \mathcal{E}_{X^*}^2(P|X^*))]}{\operatorname{Im}[\nabla : \Gamma(X^*, \mathcal{E}_{X^*}^0(P|X^*)) \longrightarrow \Gamma(X^*, \mathcal{E}_{X^*}^1(P|X^*))]}.$$

Then the inclusion

$$H^0(X, \Omega_X^1(2D)(P)) \subset \operatorname{Ker}[\nabla : \Gamma(X^*, \mathcal{E}_{X^*}^1(P|X^*)) \longrightarrow \Gamma(X^*, \mathcal{E}_{X^*}^2(P|X^*))]$$

and

$$\nabla H^0(X, \mathcal{O}_X(D)(P)) \subset \operatorname{Im}[\nabla : \Gamma(X^*, \mathcal{E}_{X^*}^0(P|X^*)) \longrightarrow \Gamma(X^*, \mathcal{E}_{X^*}^1(P|X^*))]$$

imply the existence of the desired homomorphism, and the proof of Theorem 4.2 is completed.

In his report, the referee made an interesting suggestion on another proof of Theorem 4.2, which we will explain in the rest of the present note. The referee considers a complex of sheaves over X

$$(\Omega_X^\bullet(2D)(P), \nabla) \quad : \quad \mathcal{O}_X(D)(P) \xrightarrow{\nabla} \Omega_X^1(2D)(P) \longrightarrow 0.$$

Then we have

Lemma 4.3. Two complexes of sheaves $(\Omega_X^\bullet\langle D\rangle(P), \nabla)$ and $(\Omega_X^\bullet(2D)(P), \nabla)$ are quasi-isomorphic to each other.

Proof. Obviously, we have

$$\operatorname{Ker}[\nabla : \mathcal{O}_X(P) \longrightarrow \Omega_X^1(D)(P)] \cong \operatorname{Ker}[\nabla : \mathcal{O}_X(D)(P) \longrightarrow \Omega_X^1(2D)(P)].$$

Since $\Omega_X^1(2D)(P) = \Omega_X^1(D)(P) + \operatorname{Im}[\nabla : \mathcal{O}_X(D)(P) \longrightarrow \Omega_X^1(2D)(P)]$, we have an isomorphism

$$\frac{\Omega_X^1(D)(P)}{\operatorname{Im}[\nabla : \mathcal{O}_X(P) \longrightarrow \Omega_X^1(D)(P)]} \cong \frac{\Omega_X^1(2D)(P)}{\operatorname{Im}[\nabla : \mathcal{O}_X(D)(P) \longrightarrow \Omega_X^1(2D)(P)]}.$$

Then the lemma follows immediately.

Combining this lemma with Lemma 1.2 in [1], we have the isomorphism $H^1(X^*, \mathcal{L}) \cong \mathbf{H}^1(X, \Omega_X^\bullet(2D)(P), \nabla)$. By the similar argument as in section 2 in [1], we obtain a spectral sequence $'E_r^{pq}$ (with respect to an appropriate filtration) abutting to the hypercohomology group $\mathbf{H}^1(X, \Omega_X^\bullet(2D)(P), \nabla)$. The values of $'E_1$ terms are as follows: $'E_1^{0q} = H^q(X, \mathcal{O}_X(D)(P))$, $'E_1^{1q} = H^q(X, \Omega_X^1(2D)(P))$ and $'E_1^{pq} = 0$ if $p > 1$. Let us recall the assumption $n > \max\{1, 2g-2\}$ made in [1].

Lemma 4.4. Under the assumption, we have $'E_1^{0q} = {}'E_1^{1q} = 0$ if $q > 0$.

Proof. Kodaira's vanishing theorem implies $'E_1^{1q} = 0$ $(q > 0)$. Let L be the complex line bundle defined in section 1 in [1], i.e., the tensor product of the line bundle P and the line bundle of $\mathcal{O}_X(D)$. Let K be the canonical line bundle on X. Since $n > 2g - 2$, the line bundle $L \otimes K^{-1}$ is positive. Then it follows from Theorem 2.4 in Chap. VI, [3] that $'E_1^{0q} = 0$ $(q > 0)$.

From this lemma it follows that $'E_2^{pq} = 0$ if $(p, q) \neq (1, 0)$. Therefore we have $'E_2 = {}'E_\infty$ and $\mathbf{H}^1(X, \Omega_X^\bullet(2D)(P), \nabla) \cong {}'E_2^{10} \cong \mathrm{Coker}\,[\nabla : {}'E_1^{00} \longrightarrow {}'E_1^{10}]$, which proves Theorem 4.2.

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References

[1] H. Watanabe, Twisted cohomology of a punctured Riemann surface, Kumamoto J. Math., Vol. 29 (2016), 55-63.
[2] H. Cartan and S. Eilenberg, Homological Algebra, Princeton Univ. Press, 1956.
[3] O. Wells, Differential Analysis on Complex Manifolds, 3rd ed., Springer, 2008.

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