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Note on relations among multiple zeta(-star) values II

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Abstract. We obtain new evaluations of special values of multiple polylogarithms by using a limiting case of a basic hypergeometric identity of G. E. Andrews.

1 Introduction

The objects of the present research are two types of multiple polylogarithms (MPLs for short):

$$\sum_{0<m_1<\cdots<m_n<\infty} \frac{z^{m_n-1}}{m_1^{k_1}\cdots m_n^{k_n}}, \tag{1}$$

$$\sum_{1\le m_1\le\cdots\le m_n<\infty} \frac{z^{m_n-1}}{m_1^{k_1}\cdots m_n^{k_n}}, \tag{2}$$

where $1\le n\in\mathbb{Z}$; $1\le k_i\in\mathbb{Z}$ $(i=1,\ldots,n-1)$, $2\le k_n\in\mathbb{Z}$; $z\in\mathbb{C}$ such that $|z|\le 1$. Our main interest in the present paper is the cases $z=\pm1$. For $z=1$, the multiple series (1) and (2) become the multiple zeta value $\zeta(\{k_i\}_{i=1}^n)$ (MZV for short) and the multiple zeta-star value $\zeta^\star(\{k_i\}_{i=1}^n)$ (MZSV for short), respectively, where $\{k_i\}_{i=1}^n := k_1,\ldots,k_n$ (see Euler [9], Hoffman [11], and Zagier [22]). We denote by $\zeta_-(\{k_i\}_{i=1}^n)$ and $\zeta_-^\star(\{k_i\}_{i=1}^n)$ the case $z=-1$ of (1) and of (2), respectively. Since the 1990s, MPLs have attracted much interest from many researchers because of their surprising properties. One is that their special values satisfy numerous relations (see, e.g., [4], [7], [8], [10]). This property is particularly striking for MZV and MZSV. In the present paper, we also study this property. Our main tool for the study is a hypergeometric identity. The generalized hypergeometric series is defined by the power series

$${}_{p+1}F_p\left(\begin{matrix} a_1,\ldots,a_{p+1} \\ b_1,\ldots,b_p\end{matrix};z\right) := \sum_{m=0}^{\infty} \frac{(a_1)_m\cdots(a_{p+1})_m}{(b_1)_m\cdots(b_p)_m}\frac{z^m}{m!},$$

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where $1\leq p \in \mathbb{Z}$; $z,\ a_1,\ldots,a_{p+1} \in \mathbb{C}$; $b_1,\ldots,b_p \in \mathbb{C}\setminus\{0,-1,-2,\ldots\}$; $(a)_m$ denotes the Pochhammer symbol, i.e., $(a)_m = a(a+1)\cdots(a+m-1)$ $(1\leq m \in \mathbb{Z})$ and $(a)_0 = 1$. This series converges absolutely for all $z \in \mathbb{C}$ such that $|z| = 1$ provided $\mathrm{Re}\,(\sum_{i=1}^{p} b_i - \sum_{i=1}^{p+1} a_i) > 0$. In [13] and [14], we applied the following hypergeometric identity of G. E. Andrews to studying special values of MPLs:

Theorem A (Krattenthaler and Rivoal [17, Proposition 1 (i)], a limiting case of Andrews' identity [1, Theorem 4])**.** *Let s be a positive integer and a, b_i, c_i $(i = 1,\ldots,s+1)$ complex numbers. Suppose that the complex numbers a, b_i, c_i $(i = 1,\ldots,s+1)$ satisfy the conditions $1+a-b_i$, $1+a-c_i \notin \{0,-1,-2,\ldots\}$ $(i = 1,\ldots,s+1)$,*

$$\mathrm{Re}\left((2s+1)(a+1) - 2\sum_{i=1}^{s+1}(b_i+c_i)\right) > 0,$$

$$\mathrm{Re}\left(\sum_{i=r}^{s+1} A_i(1+a-b_i-c_i)\right) > 0 \quad (r = 2,\ldots,s+1)$$

for all possible choices of $A_i = 1$ or 2 $(i = 2,\ldots,s)$, $A_{s+1} = 1$. (For details of the choices of A_i, see [17].) Then

$$\begin{aligned}
&{}_{2s+4}F_{2s+3}\left(\begin{matrix} a, \frac{a}{2}+1, b_1, c_1, \ldots, b_{s+1}, c_{s+1} \\ \frac{a}{2}, 1+a-b_1, 1+a-c_1, \ldots, 1+a-b_{s+1}, 1+a-c_{s+1}\end{matrix}; -1\right) \\
=&\frac{\Gamma(1+a-b_{s+1})\Gamma(1+a-c_{s+1})}{\Gamma(1+a)\Gamma(1+a-b_{s+1}-c_{s+1})} \\
&\times \sum_{l_1,\ldots,l_s=0}^{\infty} \prod_{i=1}^{s} \frac{(1+a-b_i-c_i)_{l_i}(b_{i+1})_{l_1+\cdots+l_i}(c_{i+1})_{l_1+\cdots+l_i}}{l_i!(1+a-b_i)_{l_1+\cdots+l_i}(1+a-c_i)_{l_1+\cdots+l_i}}.
\end{aligned}$$

As a result, we obtained various new and interesting relations among the special values, e.g., the identities (29), (30), (28), (58) of [14]; see also [13, (**A3**), (**A4**), and Addendum]. (The identities (28) and (58) of [14] were discovered in 2014.) The results of [14] show that Theorem A is very useful for this kind of research. In the present paper, we again deal with the application of Theorem A, and obtain new evaluations of special values of MPLs; see Theorems 1, 2, and 3 below. Here we note that Krattenthaler and Rivoal [17] applied Theorem A to proving an identity between hypergeometric series and multiple integrals related to construction of $\mathbb{Q}$-linear forms in the Riemann zeta values $\zeta(k) = \sum_{m=1}^{\infty} m^{-k}$, which is Zudilin's identity [24, Theorem 5]. Their prior work is on an application of Andrews' identity to diophantine approximation of zeta values. Our research on the application to special values of MPLs was begun in 2009. The papers [1] and [17] gave us a motivation of [13], [14], and of the present research. See also [12, Remarks 2.6 and 2.7] and page 2 of arXiv:0908.2536v1. We obtained Applications 1 and 2 below in April–July 2013 and June 2014, respectively. For other notes related to the present research, see [15, Introduction, Notes 1 and 2], which is a preprint of the present paper.

2 Applications of Andrews' hypergeometric identity

Hereafter we use the following notation:

$$\mathbb{Z}_{\geq k} := \{k, k+1, k+2, \ldots\}, \quad \mathbb{Z}_{\leq k} := \{k, k-1, k-2, \ldots\},$$
$$\{a\}^n := \underbrace{a, \ldots, a}_{n},$$

where $k \in \mathbb{Z}$ and $1 \leq n \in \mathbb{Z}$. We regard $\{a\}^0$ as the empty set $\emptyset$.

2.1 Application 1

Using Theorem A, we prove the following evaluations:

Theorem 1. *Let $r \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{Z}_{\geq 1}$. Then the sums*

$$\sum_{i=0}^{r} (-1)^{r-i} \left\{ \binom{s-2+i}{i} + \binom{s-1+i}{i} \right\} \zeta_-(\{1\}^{r-i}, 2s+i), \tag{3}$$

$$\sum_{i=0}^{r} (-1)^{r-i} 2^{r-i+1} \left\{ \binom{2s+i-1}{i} - \binom{2s+i-1}{i-1} \right\} \zeta_-(\{1\}^{r-i}, 2s+i) \tag{4}$$

can be expressed in $\mathbb{Q}$-polynomials of the Riemann zeta values $\zeta(k) = \sum_{m=1}^{\infty} m^{-k}$ ($k \in \mathbb{Z}_{\geq 2}$).

Remark 1. We first note the following two evaluations:

$$\begin{aligned} \zeta_-(\{1\}^m, 2) =& (-1)^m \zeta(m+2) + (-1)^m 2 \frac{(\log 2)^{m+2}}{(m+2)!} \\ &+ (-1)^{m+1} \sum_{k=0}^{m+2} \mathrm{Li}_k(1/2) \frac{(\log 2)^{m+2-k}}{(m+2-k)!} \end{aligned} \tag{5}$$

($m \in \mathbb{Z}_{\geq 0}$; Borwein, Bradley, and Broadhurst [6, Identity (69), Section 6]), where $\mathrm{Li}_k(z) := \sum_{m=1}^{\infty} z^m m^{-k}$ is the polylogarithm; and

$$\begin{aligned} \zeta_-(1,3) =& -2\mathrm{Li}_4(1/2) - \frac{1}{12}(\log 2)^4 + \frac{15}{8}\zeta(4) \\ &- \frac{7}{4}\zeta(3)\log 2 + \frac{1}{2}\zeta(2)(\log 2)^2 \end{aligned} \tag{6}$$

(Borwein, Borwein, and Girgensohn [5, p. 291, line 10]). We note that, while the evaluations (5) and (6) contain $\mathrm{Li}_k(1/2)$ ($k \in \mathbb{Z}_{\geq 1}$, $\log 2 = \mathrm{Li}_1(1/2)$), the evaluations of the sums (3) and (4) in Theorem 1 do not contain $\mathrm{Li}_k(1/2)$. In fact, they can be evaluated only by $\zeta(k)$ ($= \mathrm{Li}_k(1)$). From this fact, we think that the sums (3) and (4) are particularly interesting. Each of $\zeta_-(\{1\}^m, n+1)$'s ($m, n \in \mathbb{Z}_{\geq 0}$) can be evaluated by alternating Euler sums (see Borwein et al. [6, Identity (28)] and [7, Theorem 9.3]). For MZVs, it is known that $\zeta(\{1\}^m, n+2)$ ($m, n \in \mathbb{Z}_{\geq 0}$) can be expressed in a $\mathbb{Q}$-polynomial of $\zeta(k)$ (see [20] and [22]).

To prove Theorem 1, we need two lemmas. The first gives expressions of (3) and (4) as sums of MZSVs:

Lemma 1. *We have*

(i)

$$\begin{aligned}&\sum_{i=0}^{r}(-1)^{r-i}\left\{\binom{s-2+i}{i}+\binom{s-1+i}{i}\right\}\zeta_-(\{1\}^{r-i},2s+i)\\&=\sum_{\substack{r_1+\cdots+r_s=r\\ r_i\in\mathbb{Z}_{\geq 0}}}\zeta^\star(\{r_i+2\}_{i=1}^{s})\end{aligned}\tag{7}$$

for all $r\in\mathbb{Z}_{\geq 0}$, $s\in\mathbb{Z}_{\geq 1}$.

(ii)

$$\begin{aligned}&\sum_{i=0}^{r}(-1)^{r-i}2^{r-i+1}\left\{\binom{2s+i-1}{i}-\binom{2s+i-1}{i-1}\right\}\zeta_-(\{1\}^{r-i},2s+i)\\&=\sum_{i=0}^{r}\frac{(-1)^i}{i!}\frac{\mathrm{d}^i}{\mathrm{d}\alpha^i}\left(\frac{\Gamma(\alpha)^2}{\Gamma(2\alpha-1)}\right)\Big|_{\alpha=1}\\&\times\sum_{\substack{r_1+\cdots+r_s=r-i\\ r_j\in\mathbb{Z}_{\geq 0}}}\left\{\prod_{j=1}^{s}(r_j+1)\right\}\zeta^\star(\{r_j+2\}_{j=1}^{s})\end{aligned}\tag{8}$$

for all $r\in\mathbb{Z}_{\geq 0}$, $s\in\mathbb{Z}_{\geq 1}$, *where* $\Gamma(z)$ *is the gamma function and* $\binom{2s+i-1}{i-1}=0$ *if* $i=0$.

Proof. Taking $\alpha=\beta=1$ in [14, Theorem 2.13 (iii)], we have (7). The proof of (8) is as follows: Taking $a=2\alpha$ and $b_i=c_i=\alpha$ $(i=1,\ldots,s+1)$ $(s\in\mathbb{Z}_{\geq 1}$, $\alpha\in\mathbb{C}$ with $s+1/2>\mathrm{Re}(\alpha)>1/2)$ in Theorem A, we have

$$\begin{aligned}&2\sum_{m=0}^{\infty}\frac{(2\alpha-1)_{m+1}}{m!}\frac{(-1)^m}{(m+\alpha)^{2s+1}}\\&=\frac{\Gamma(\alpha)^2}{\Gamma(2\alpha-1)}\sum_{0\leq m_1\leq\cdots\leq m_s<\infty}\prod_{i=1}^{s}\frac{1}{(m_i+\alpha)^2}.\end{aligned}$$

Differentiating both sides of this identity r times at $\alpha=1$ and using the identities

$$\begin{aligned}\frac{1}{r!}\frac{\mathrm{d}^r}{\mathrm{d}w^r}(w)_{m+1}=&(w)_{m+1}\sum_{0\leq m_1<\cdots<m_r\leq m}\prod_{i=1}^{r}\frac{1}{m_i+w}\\=&(w)_{m+1}\left(\sum_{0\leq m_1<\cdots<m_r<m}+\sum_{0\leq m_1<\cdots<m_r=m}\right)\prod_{i=1}^{r}\frac{1}{m_i+w}\end{aligned}$$

$(r,m\in\mathbb{Z}_{\geq 0})$, we obtain (8).

The second lemma gives evaluations of the sums of MZSVs in Lemma 1 in terms of $\zeta(k)$:

Lemma 2. *Let $k \in \mathbb{Z}_{\geq 1}$, $q, r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 2}$, and let $f(r_1, \ldots, r_k)$ be a symmetric function with k variables. Then the sum*

$$\sum_{\substack{r_1+\cdots+r_k=r \\ r_i \in \mathbb{Z}_{\geq 0}}} f(r_1, \ldots, r_k) \zeta^\star(qr_1+s, \ldots, qr_k+s) \tag{9}$$

can be expressed in a $\mathbb{Q}$-linear combination of $f(r_1, \ldots, r_k)\{\prod_{i=1}^{\ell} \zeta(m_i)\}$ ($m_i \in \mathbb{Z}_{\geq 2}$).

Proof. Let $\mathfrak{S}_k$ be the symmetric group of degree k. Then we have

$$\begin{aligned} &\sum_{\sigma \in \mathfrak{S}_k} \sum_{\substack{r_{\sigma(1)}+\cdots+r_{\sigma(k)}=r \\ r_{\sigma(i)} \in \mathbb{Z}_{\geq 0}}} f(r_{\sigma(1)}, \ldots, r_{\sigma(k)}) \zeta^\star(qr_{\sigma(1)}+s, \ldots, qr_{\sigma(k)}+s) \\ &= \sum_{\substack{r_1+\cdots+r_k=r \\ r_i \in \mathbb{Z}_{\geq 0}}} f(r_1, \ldots, r_k) \sum_{\sigma \in \mathfrak{S}_k} \zeta^\star(qr_{\sigma(1)}+s, \ldots, qr_{\sigma(k)}+s) \end{aligned} \tag{10}$$

for $k \in \mathbb{Z}_{\geq 1}$, $q, r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 2}$. We examine each side of (10). Taking $i_j = qr_j+s$ ($j = 1, \cdots, k$) in [11, Theorem 2.1], we see that the inner sum on the right-hand side of (10) can be written as a $\mathbb{Q}$-polynomial of $\zeta(m)$ ($m \in \mathbb{Z}_{\geq 2}$). Thus the right-hand side becomes a $\mathbb{Q}$-linear combination of $f(r_1, \ldots, r_k)\{\prod_{i=1}^{\ell} \zeta(m_i)\}$. On the other hand, since $|\mathfrak{S}_k| = k!$, where $|\mathfrak{S}_k|$ denotes the number of elements of $\mathfrak{S}_k$, the left-hand side of (10) can be rewritten as

$$k! \sum_{\substack{r_1+\cdots+r_k=r \\ r_i \in \mathbb{Z}_{\geq 0}}} f(r_1, \ldots, r_k) \zeta^\star(qr_1+s, \ldots, qr_k+s),$$

which is (9). This completes the proof.

Remark 2. Hoffman proved his evaluation [11, Theorem 2.1] explicitly; therefore the sum (9) can also be evaluated like that.

Proof of Theorem 1. The case $q = 1$, $s = 2$, $f(r_1, \ldots, r_k) = 1$ of Lemma 2 shows that the right-hand side of (7) can be expressed in a $\mathbb{Q}$-linear combination of $\prod_{i=1}^{\ell} \zeta(m_i)$, and this gives a proof of the assertion for (3). Similarly, the case $q = 1$, $s = 2$, $f(r_1, \ldots, r_k) = \prod_{i=1}^{k}(r_i+1)$ of Lemma 2 shows that the inner sum on the right-hand side of (8) can be expressed in a $\mathbb{Q}$-linear combination of $\{\prod_{i=1}^{k}(r_i+1)\}\{\prod_{i=1}^{\ell} \zeta(m_i)\}$. As regards the differential coefficient

$$\frac{1}{i!} \frac{\mathrm{d}^i}{\mathrm{d}\alpha^i} \left(\frac{\Gamma(\alpha)^2}{\Gamma(2\alpha-1)} \right) \Big|_{\alpha=1} \qquad (i \in \mathbb{Z}_{\geq 0}), \tag{11}$$

it also has such an expression. Indeed, using the expansion of the gamma function

$$\Gamma(\alpha) = \exp\left(-\gamma(\alpha-1) + \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} (\alpha-1)^n \right)$$

for all $\alpha \in \mathbb{C}$ such that $|\alpha - 1| < 1$, where γ is Euler's constant (see, e.g., [3, p. 38], [21, Chapter XII], and [20]), we have

$$\frac{\Gamma(\alpha)^2}{\Gamma(2\alpha-1)} = \exp\left(\sum_{n=2}^{\infty}(-1)^{n+1}\frac{\zeta(n)}{n}(2^n-2)(\alpha-1)^n\right)$$

for $|\alpha - 1| < 1/2$, and this gives an expression of (11) by a $\mathbb{Q}$-polynomial of $\zeta(k)$. Therefore, combining all the above expressions of the right-hand side of (8), we obtain a proof of the assertion for (4).

Using Theorem A, we can prove also the following evaluation, which is similar to (7):

Theorem 2. *We have*

$$\begin{aligned}&\sum_{i=0}^{r}\left\{\binom{s-2+i}{i}+\binom{s-1+i}{i}\right\}\zeta_{-}^{\star}(\{1\}^{r-i},2s+i)\\ &=\sum_{\substack{r_1+\cdots+r_s=r\\ r_i\in\mathbb{Z}_{\geq 0}}}(r_1+1)\zeta^{\star}(\{r_i+2\}_{i=1}^{s})\end{aligned} \tag{12}$$

for all $r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$.

Proof. Taking $a = \alpha + 1$, $b_1 = c_1 = 1$, $b_i = \alpha$, $c_i = 1$ $(i = 2, \ldots, s+1)$ $(s \in \mathbb{Z}_{\geq 1}$, $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0)$ in Theorem A, we have

$$\begin{aligned}&\sum_{m=0}^{\infty}\frac{m!}{(\alpha)_{m+1}}\frac{2m+\alpha+1}{(m+\alpha)^s(m+1)^s}(-1)^m\\ &=\sum_{0\leq m_1\leq\cdots\leq m_s<\infty}\frac{1}{(m_1+\alpha)^2}\left\{\prod_{i=2}^{s}\frac{1}{(m_i+\alpha)(m_i+1)}\right\}.\end{aligned}$$

Differentiating both sides of this identity r times at $\alpha = 1$ and using the identity

$$\frac{(-1)^r}{r!}\frac{\mathrm{d}^r}{\mathrm{d}w^r}\left(\frac{1}{(w)_{m+1}}\right) = \frac{1}{(w)_{m+1}}\sum_{0\leq m_1\leq\cdots\leq m_r\leq m}\prod_{i=1}^{r}\frac{1}{m_i+w}$$

$(r, m \in \mathbb{Z}_{\geq 0})$, we obtain (12).

Remark 3. (i) Taking $\alpha = \beta = \gamma = 1$ in [14, Theorem 2.13 (ii)], we have another expression of the left-hand side of (12):

$$\text{The left-hand side of (12)} = \sum_{\substack{r_1+\cdots+r_s=r\\ r_i\in\mathbb{Z}_{\geq 0}}}\zeta^{\star}(\{\{1\}^{r_i},2\}_{i=1}^{s})$$

$(r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1})$, where $\{\{1\}^{r_i}, 2\}_{i=1}^{s} := \{1\}^{r_1}, 2, \ldots, \{1\}^{r_i}, 2, \ldots, \{1\}^{r_s}, 2.$

(ii) For $s = 1$, the identities (7) and (12) become

$$\sum_{i=0}^{r} (-1)^{r-i}(\delta_{i0} + 1)\zeta_{-}(\{1\}^{r-i}, 2 + i) = \zeta(r + 2), \tag{13}$$

$$\sum_{i=0}^{r} (\delta_{i0} + 1)\zeta_{-}^{\star}(\{1\}^{r-i}, 2 + i) = (r + 1)\zeta(r + 2) \tag{14}$$

($r \in \mathbb{Z}_{\geq 0}$), respectively, where δ_{ij} denotes Kronecker's delta, i.e., $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. We note that there is a similarity between (13), (14) and the identity for MZSVs of Aoki and Ohno [2, Corollary 1].

2.2 Application 2

As shown in [14], the following identity can be derived from Theorem A:

$$\begin{aligned} &\sum_{i=0}^{k} 2^{k-i} \sum_{\substack{k_1+\cdots+k_s=k-i \\ k_j \in \mathbb{Z}_{\geq 0}}} \zeta^{\star}(i + k_1 + 2, \{k_j + 2\}_{j=2}^{s}) \\ =&2^{1+k}\binom{k+s-1}{k}(1 - 2^{1-k-2s})\zeta(k + 2s) \end{aligned} \tag{15}$$

for all $k \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$. (For details, see [14, Theorem 2.16 and its proof, Remark 8].) Using this identity, we can prove the following evaluation of MZSVs:

Theorem 3. *We have*

$$\begin{aligned} &\zeta^{\star}(4, \{2\}^{s-1}) \\ =&c(s)\zeta^{\star}(\{2\}^{s+1}) - \frac{2}{3} \sum_{\substack{k_1+k_2+k_3=s-2 \\ k_j \in \mathbb{Z}_{\geq 0}}} (2 + \delta_{0k_1})\zeta^{\star}(\{2\}^{k_1}, 3, \{2\}^{k_2}, 3, \{2\}^{k_3}) \end{aligned} \tag{16}$$

for all $s \in \mathbb{Z}_{\geq 1}$, where δ_{ij} denotes Kronecker's delta defined under (14) *and*

$$c(s) := \frac{2s(s+1)}{3} + \frac{2}{3}\sum_{i=2}^{s+1}\binom{2(s+1)}{2i}\frac{B_{2(s+1-i)}B_{2i}}{B_{2(s+1)}}\frac{(1 - 2^{1-2i})}{(1 - 2^{1-2(s+1)})} \tag{17}$$

($s \in \mathbb{Z}_{\geq 1}$). *Here B_n is the n-th Bernoulli number.*

Remark 4. The identity (16) is an expression of $\zeta^{\star}(4, \{2\}^{s-1})$ in terms of the $\{2, 3\}$-basis of the $\mathbb{Q}$-vector space of MZSVs. For the $\{2, 3\}$-basis, see [16, Section 3].

Proof of Theorem 3. To prove (16), we use the case $k = 2$ of (15):

$$\begin{aligned} &3\zeta^{\star}(4, \{2\}^{s-1}) + 2^2 \sum_{i=1}^{s} \zeta^{\star}(\{2\}^{i-1}, 4, \{2\}^{s-i}) \\ &+ 2 \sum_{\substack{k_1+k_2+k_3=s-2 \\ k_j \in \mathbb{Z}_{\geq 0}}} (2 + \delta_{0k_1})\zeta^{\star}(\{2\}^{k_1}, 3, \{2\}^{k_2}, 3, \{2\}^{k_3}) \\ =&2^2 s(s + 1)(1 - 2^{-1-2s})\zeta(2 + 2s) \end{aligned} \tag{18}$$

($s \in \mathbb{Z}_{\geq 1}$). By using the identity

$$2(1-2^{1-2s})\zeta(2s) = \zeta^{\star}(\{2\}^{s}) \tag{19}$$

($s \in \mathbb{Z}_{\geq 1}$; see [23]), the right-hand side of (18) can be rewritten as

$$2^{2}s(s+1)(1-2^{-1-2s})\zeta(2+2s) = 2s(s+1)\zeta^{\star}(\{2\}^{s+1}) \tag{20}$$

for $s \in \mathbb{Z}_{\geq 1}$. As regards the first sum on the left-hand side of (18), by using (19) and Euler's formula

$$\zeta(2s) = (-1)^{s-1}\frac{B_{2s}}{(2s)!}\frac{(2\pi)^{2s}}{2} \qquad (s \in \mathbb{Z}_{\geq 1}),$$

it can be rewritten as

$$\begin{aligned}
&\sum_{i=1}^{s}\zeta^{\star}(\{2\}^{i-1},4,\{2\}^{s-i}) \\
=&\sum_{i=0}^{s-1}\zeta(2(i+2))\zeta^{\star}(\{2\}^{s-1-i}) \qquad \text{(see the proof below)} \\
=&2\sum_{i=0}^{s-1}(1-2^{1-2(s-1-i)})\zeta(2(i+2))\zeta(2(s-1-i)) \\
=&\left(\sum_{i=0}^{s-1}(1-2^{1-2(s-1-i)})\frac{B_{2(i+2)}B_{2(s-1-i)}}{(2(i+2))!(2(s-1-i))!}\right)(-1)^{s-1}\frac{(2\pi)^{2(s+1)}}{2} \\
=&-\left(\sum_{i=0}^{s-1}(1-2^{1-2(s-1-i)})\frac{B_{2(i+2)}B_{2(s-1-i)}}{(2(i+2))!(2(s-1-i))!}\right)\frac{(2(s+1))!}{B_{2(s+1)}}\zeta(2(s+1)) \\
=&-\frac{1}{2}\left(\sum_{i=0}^{s-1}\binom{2(s+1)}{2(i+2)}\frac{B_{2(i+2)}B_{2(s-1-i)}}{B_{2(s+1)}}\frac{(1-2^{1-2(s-1-i)})}{(1-2^{1-2(s+1)})}\right)\zeta^{\star}(\{2\}^{s+1})
\end{aligned} \tag{21}$$

for $s \in \mathbb{Z}_{\geq 1}$, where $\zeta(0) = -1/2$ and $\zeta^{\star}(\emptyset) = 1$. Therefore, substituting (20) and (21) into (18), we obtain (16). The first identity of (21) can be proved in a way similar to used for MZVs in [3, pp. 8 and 89]. Indeed, using the harmonic product of MZSVs (see [16] and [18]), we have

$$\begin{aligned}
\zeta(t)\zeta^{\star}(\{2\}^{s-1}) =&\sum_{i=1}^{s}\zeta^{\star}(\{2\}^{i-1},t,\{2\}^{s-i}) - \sum_{i=1}^{s-1}\zeta^{\star}(\{2\}^{i-1},t+2,\{2\}^{s-1-i}), \\
\zeta(t+2)\zeta^{\star}(\{2\}^{s-2}) =&\sum_{i=1}^{s-1}\zeta^{\star}(\{2\}^{i-1},t+2,\{2\}^{s-1-i}) - \sum_{i=1}^{s-2}\zeta^{\star}(\{2\}^{i-1},t+4, \\
&\{2\}^{s-2-i}), \\
&\vdots \\
\zeta(t+2s-4)\zeta^{\star}(2) =&\zeta^{\star}(t+2s-4,2) + \zeta^{\star}(2,t+2s-4) - \zeta(t+2s-2)
\end{aligned}$$

for $s, t \in \mathbb{Z}_{\geq 2}$. Further, adding up each side of all these identities, we obtain

$$\sum_{i=0}^{s-1} \zeta(2i+t)\zeta^\star(\{2\}^{s-1-i}) = \sum_{i=1}^{s} \zeta^\star(\{2\}^{i-1}, t, \{2\}^{s-i})$$

for $s \in \mathbb{Z}_{\geq 1}$, $t \in \mathbb{Z}_{\geq 2}$. The first identity of (21) is the case $t = 4$ of this identity. This completes the proof of Theorem 3.

Remark 5. (i) The identity (15) contains both (19) and the identity for $\zeta(2s+1)$ of [16, Theorem 2] as special cases: $k = 0$ and $k = 1$, respectively. Compare this with [16, Proof of Theorem 2]. These two cases give a $\{2,3\}$-basis expression of $\zeta(k)$ for any $k \in \mathbb{Z}_{\geq 2}$ (see [16, Theorem 2]).

(ii) Taking $\alpha = 1$ in [14, Theorem 2.16], we have the following general form of (15):

$$\begin{aligned}
&\sum_{i=0}^{k} 2^{k-i}\binom{i+r}{i} \sum_{\substack{k_1+\cdots+k_s=k-i\\ k_j\in\mathbb{Z}_{\geq 0}}} \zeta^\star(i+k_1+r+2, \{k_j+2\}_{j=2}^{s})\\
=&\sum_{i=0}^{r} \sum_{\substack{k_1+\cdots+k_{i+2}=k\\ r_1+\cdots+r_{i+1}=r+1\\ k_j\in\mathbb{Z}_{\geq 0}, r_j\in\mathbb{Z}_{\geq 1}}} 2^{i+1+k_{i+2}} \left\{\prod_{j=1}^{i}\binom{k_j+r_j-1}{k_j}\right\}\binom{k_{i+1}+r_{i+1}-2}{k_{i+1}}\\
&\times\binom{k_{i+2}+s-1}{k_{i+2}}\zeta_-(\{k_j+r_j\}_{j=1}^{i}, k_{i+1}+k_{i+2}+r_{i+1}+2s-1)
\end{aligned}$$

for all $k, r \in \mathbb{Z}_{\geq 0}$, $s \in \mathbb{Z}_{\geq 1}$. It is interesting to find applications of this general form, and also to obtain generalizations of (16) concerning the $\{2,3\}$-basis expression of MZSVs.

Remark 6. (i) This was obtained in 2018. The coefficient (17) has the following closed form:

$$\begin{aligned}
c(s) =&\frac{2}{3}\{s(s-1)-1+(1-2^{1-2(s+1)})^{-1}\}\\
&-\frac{1}{18}\frac{(s+1)(2s+1)}{(1-2^{1-2(s+1)})}\frac{B_{2s}}{B_{2(s+1)}}
\end{aligned} \tag{22}$$

for all $s \in \mathbb{Z}_{\geq 1}$. Indeed, taking $n = s+1$ in the identity on page 154, line 11 of [19] and using $B_0 = 1$, $B_2 = 1/6$, we have the following identity for the Bernoulli numbers:

$$\begin{aligned}
&\sum_{i=2}^{s+1}\binom{2(s+1)}{2i} B_{2(s+1-i)}B_{2i}(1-2^{1-2i})\\
=&\{1-(2s+1)(1-2^{1-2(s+1)})\}B_{2(s+1)} - \frac{(s+1)(2s+1)}{12}B_{2s}
\end{aligned}$$

for $s \in \mathbb{Z}_{\geq 1}$. Dividing both sides of this identity by $B_{2(s+1)}(1-2^{1-2(s+1)})$ and substituting the result into (17), we obtain (22).

(ii) In [14, Remark 7 (i)], we gave a new proof of Hoffman's identity $\zeta(\{1\}^k, l+2) = \zeta(\{1\}^l, k+2)$ ($k, l \in \mathbb{Z}_{\geq 0}$; [11, Theorem 4.4]), which is the duality formula for $\zeta(\{1\}^k, l+2)$. Our proof is based on the hypergeometric identities [1, Theorem 4] and [17, Proposition 1 (i)]; therefore it can be regarded as a hypergeometric proof of the duality formula. It is interesting to generalize our proof to a proof of the duality formula for all MZVs in appropriate ways.

Corrections to [14]. (i) Page 713, line 9 from the bottom: "and a revised" should be "and is a revised". (ii) Page 726, line 10 from the bottom: "I remark that" should be "I note that". (iii) Page 726, line 8 from the bottom: "and my observation" should be "and from my observation". (iv) Page 755, lines 3–4: "manuscript (2013)." should be "manuscript, submitted to a journal on March 5, 2013 and rejected on May 9, 2013.".

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References

[1] G. E. Andrews, Problems and prospects for basic hypergeometric functions, in *Theory and application of special functions*, Ed. by R. A. Askey, Math. Res. Center, Univ. Wisconsin, Publ. No. 35, Academic Press, 1975, New York, pp. 191–224.

[2] T. Aoki and Y. Ohno, Sum relations for multiple zeta values and connection formulas for the Gauss hypergeometric functions, Publ. Res. Inst. Math. Sci. **41** (2005), no. 2, 329–337.

[3] T. Arakawa and M. Kaneko, *Introduction to multiple zeta values*, Kyushu University MI Lecture Note Series, Vol. 23 (2010) (in Japanese).

[4] J. Blümlein, D. J. Broadhurst and J. A. M. Vermaseren, The multiple zeta value data mine, Comput. Phys. Comm. **181** (2010), no. 3, 582–625.

[5] D. Borwein, J. M. Borwein and R. Girgensohn, Explicit evaluation of Euler sums, Proc. Edinburgh Math. Soc. (2) **38** (1995), no. 2, 277–294.

[6] J. M. Borwein, D. M. Bradley and D. J. Broadhurst, Evaluations of k-fold Euler/Zagier sums: a compendium of results for arbitrary k, The Wilf Festschrift (Philadelphia, PA, 1996), Electron. J. Combin. **4** (1997), no. 2, Research Paper 5, approx. 21 pp. (electronic).

[7] J. M. Borwein, D. M. Bradley, D. J. Broadhurst and P. Lisoněk, Special values of multiple polylogarithms, Trans. Amer. Math. Soc. **353** (2001), no. 3, 907–941.

[8] D. Bowman and D. M. Bradley, Multiple polylogarithms: a brief survey, in *q-series with applications to combinatorics, number theory, and physics* (Urbana, IL, 2000), 71–92, Contemp. Math., **291**, Amer. Math. Soc., Providence, RI, 2001.

[9] L. Euler, Meditationes circa singulare serierum genus, Novi Comm. Acad. Sci. Petropol. **20** (1775), 140–186; reprinted in Opera Omnia, Ser. I, Vol. 15, B. G. Teubner, Berlin, 1927, pp. 217–267.

[10] A. B. Goncharov, Multiple polylogarithms and mixed Tate motives, 2001. arXiv:math/0103059v4.

[11] M. E. Hoffman, Multiple harmonic series, Pacific J. Math. **152** (1992), no. 2, 275–290.

[12] M. Igarashi, Cyclic sum of certain parametrized multiple series, J. Number Theory **131** (2011), no. 3, 508–518.

[13] M. Igarashi, Note on relations among multiple zeta-star values, 2011. http://arxiv.org/abs/1106.0481.

[14] M. Igarashi, Note on relations among multiple zeta(-star) values, Italian J. Pure Appl. Math. no. 39 (2018), 710–756; submitted on July 9, 2016.

[15] M. Igarashi, Note on relations among multiple zeta(-star) values II, 2022. http://arxiv.org/abs/2007.11873v18.

[16] K. Ihara, J. Kajikawa, Y. Ohno and J. Okuda, Multiple zeta values vs. multiple zeta-star values, J. Algebra **332** (2011), no. 1, 187–208.

[17] C. Krattenthaler and T. Rivoal, An identity of Andrews, multiple integrals, and very-well-poised hypergeometric series, Ramanujan J. **13** (2007), no. 1-3, 203–219.

[18] S. Muneta, Algebraic setup of non-strict multiple zeta values, Acta Arith. **136** (2009), no. 1, 7–18.

[19] T. Nakamura, Restricted and weighted sum formulas for double zeta values of even weight, Šiauliai Math. Semin. **4** (12) (2009), 151–155.

[20] Y. Ohno and D. Zagier, Multiple zeta values of fixed weight, depth and height, Indag. Math. (N. S.) **12** (2001), no. 4, 483–487.

[21] E. T. Whittaker and G. N. Watson, *A Course of Mordern Analysis*, 4th ed., Cambridge Univ. Press, Cambridge, UK, 1927; reprinted 2003.

[22] D. Zagier, Values of zeta functions and their applications, in *First European Congress of Mathematics*, Vol. II (Paris, 1992), Ed. by A. Joseph et al., Progr. Math. **120**, Birkhäuser, 1994, Basel, pp. 497–512.

[23] S. A. Zlobin, Generating functions for the values of a multiple zeta function, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **60** (2005), no. 2, 55–59 (in Russian); English transl., Moskow Univ. Math. Bull. **60** (2005), no. 2, 44–48.

[24] W. Zudilin, Well-poised hypergeometric service for diophantine problems of zeta values, J. Théor. Nombres Bordeaux **15** (2003), no. 2, 593–626.

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