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Contiguity operators for the classical hypergeometric functions and their Lie algebra structure

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Abstract. We discuss the derivation of the explicit form of contiguity relations and operators for the Gauss hypergeometric function and for its confluent family from the view point of Gelfand's hypergeometric function. By this approach, Lie algebraic nature of the contiguity relations is made apparent and derivation of them becomes in a unified way for the Gauss hypergeometric and its confluent family, which was originally discussed case by case.

1 Introduction

In this paper, we revisit the contiguity relations for the classical hypergeometric functions (HGF, for short) from the view point of Gelfand's HGF on the Grassmannian manifold and make clear the Lie algebraic structure of the set of contiguity operators. The classical HGF we consider are the Gauss HGF and its confluent family, namely Kummer's confluent HGF, Bessel function, Hermite-Weber function and Airy function. Each function is characterized by the second order linear differential equation on the complex projective space $\mathbb{P}^1$. Let x be the affine coordinate of $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Then the equations are

$$
\begin{aligned}
\text{Gauss:} \quad & x(1-x)y'' + \{c - (a+b+1)x\}y' - aby = 0, \\
\text{Kummer:} \quad & xy'' + (c-x)y' - au = 0, \\
\text{Bessel:} \quad & xy'' + (c+1)y' + y = 0, \\
\text{Hermite-Weber:} \quad & y'' - xy' + ay = 0, \\
\text{Airy:} \quad & y'' - xy = 0.
\end{aligned}
$$

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These equations are denoted as $E_G, E_K, E_B, E_{HW}, E_A$, respectively. Among them, Gauss, Kummer and Bessel equations have a regular singular point at $x = 0$ and have the solutions expressed by the power series

$$\begin{aligned} {}_2F_1(a,b,c;x) &= \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m m!} x^m, \\ {}_1F_1(a,c;x) &= \sum_{m=0}^{\infty} \frac{(a)_m}{(c)_m m!} x^m, \\ {}_0F_1(c+1;-x) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{(c+1)_m m!} x^m, \end{aligned}$$

respectively, where

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = \begin{cases} 1, & m = 0, \\ a(a+1)\cdots(a+m-1), & m \geq 1 \end{cases}$$

is so-called the Pochhammer symbol expressed in terms of the gamma function $\Gamma(a)$. Let us explain what is the contiguity relation for the Gauss HGF. It is well known ([8]) that there are the first order differential operators which connect ${}_2F_1(a,b,c;x)$ to ${}_2F_1$ with one of the parameters a, b, c increased or decreased by 1. For example, we know

$$\begin{gathered} (x\partial + a)\, {}_2F_1(a,b,c;x) = a \cdot {}_2F_1(a+1,b,c;x), \\ (x(1-x)\partial + c - a - bx)\, {}_2F_1(a,b,c;x) = (c-a) \cdot {}_2F_1(a-1,b,c;x), \end{gathered}$$

where $\partial = \frac{d}{dx}$. These identities are called contiguity relations and the differential operators which give these relations are called contiguity operators. Note that these contiguity relations hold not only for ${}_2F_1(a,b,c;x)$ but for any solution of the Gauss hypergeometric equation $E_G(a,b,c)$. Fix any point $x_0 \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and let $S(a,b,c)$ denote the solution space of $E_G(a,b,c)$ at x_0. Then the differential operators $L(a^+, b, c) := x\partial + a$ and $L(a^-, b, c) := x(1-x)\partial + c - a - bx$ give linear maps

$$L(a^+, b, c) : S(a,b,c) \to S(a+1,b,c), \quad L(a^-, b, c) : S(a,b,c) \to S(a-1,b,c)$$

and when $a(c-a-1) \neq 0$, the operator $L(a^+, b, c)$ gives an isomorphism.

In this paper we derive the contiguity relations and contiguity operators explicitly for the Gauss HGF and its confluent family using the viewpoint of Gelfand's HGF. For the Gauss HGF, the contiguity relations are well studied ([3, 4, 8]) and, before the work of I. M. Gelfand ([5]), W. Miller ([14]) clarified the Lie algebra structure of the contiguity operators not only for the Gauss but also for its generalizations to several variables. Several authors studied the explicit form of contiguity relations using the viewpoint of Gelfand's HGF ([6, 12, 16]). For the confluent family of Gauss HGF, the contiguity relations are classically known, see

[3, 4] for example. The Lie algebraic aspect of contiguity relations for the classical HGF of confluent type is discussed case by case in [15] by W. Miller. On the other hand, the approach from the viewpoint of Gelfand's HGF was discussed in [10]. But in that paper, the way of obtaining the explicit form of contiguity relations was not sufficiently discussed for the Gauss's confluent family. In this paper we carry out this task for Kummer, Bessel, Hermite-Weber functions. We also demonstrate that the famous formulae $\Gamma(a+1) = a\Gamma(a)$ for the gamma function and $B(a+1,b) = \frac{a}{a+b}B(a,b)$ for the beta function are also derived by using the same idea.

This paper is organized as follows. In Section 2, we explain how the classical HGFs are connected with Gelfand's HGF (Subsection 2.3) and how the contiguity relations for Gelfand's HGF (of confluent type) are derived from the generalized root space decomposition of the Lie algebra $\mathfrak{gl}(N)$ (Subsection 2.2). In Section 3, using the facts explained in Section 2, we derive the explicit form of contiguity relations (operators) for Kummer, Bessel, Hermite-Weber functions. The results are given in Propositions 3.1, 3.3, 3.4 and 3.5. In Section 4, we show that the famous recurrence relations for the beta and the gamma functions are also derived in the same spirit.

2 Gelfand's HGF and Gauss' confluent family

In this section we review the connection of the classical HGF family with Gelfand's HGF. The key point is the integral representation of solutions of the differential equations $E_G, E_K, E_B, E_{HW}, E_A$. They are

$$
\begin{aligned}
I_G(x) &= \int_C u^{a-1}(1-u)^{c-a-1}(1-xu)^{-b}du, \\
I_K(x) &= \int_C e^{xu}u^{a-1}(1-u)^{c-a-1}du, \\
I_B(x) &= \int_C e^{u-\frac{x}{u}}u^{-c-1}dt = \int_{C'} e^{xu-\frac{1}{u}}u^{c-1}du, \\
I_{HW}(x) &= \int_C e^{xu-\frac{1}{2}u^2}u^{-a-1}du, \\
I_A(x) &= \int_C e^{xu-u^3/3}du. \qquad (2.1)
\end{aligned}
$$

If we take various paths of integration C in the integral representation $I_J(x)$, we have solutions for the differential equations E_J. For example, if we consider $I_G(x)$ and $I_K(x)$ and take the path $\overrightarrow{0,1}$ in the complex u-plain which starts from 0 and end at 1 in the integrals, then we have the power series solutions

$$
{}_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 u^{a-1}(1-u)^{c-a-1}(1-xu)^{-b}du, \qquad (2.2)
$$

$$
{}_1F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{xu}u^{a-1}(1-u)^{c-a-1}du. \qquad (2.3)
$$

It is known ([9, 11]) that these functions are identified with Gelfand's HGF on the Grassmannian $\mathrm{Gr}(2,4)$, which we shall explain below.

2.1 Gelfand's HGF

Let $N \geq 3$ be a positive integer and let $\mathrm{Gr}(2,N)$ be the Grassmannian manifold of 2 dimensional linear subspaces in $\mathbb{C}^N$. Let λ be a partition of N, namely $\lambda = (n_1, n_2, \dots, n_\ell)$ is a tuple of non-increasing integers $n_1 \geq n_2 \geq \cdots \geq n_\ell > 0$ with $|\lambda| := n_1 + n_2 + \cdots + n_\ell = N$. For example, the partitions of 4 are given by

$$(1,1,1,1),\ (2,1,1),\ (2,2),\ (3,1),\ (4). \tag{2.4}$$

To a partition λ of N we associate a maximal abelian subgroup $H_\lambda = J(n_1) \times \cdots \times J(n_\ell)$ of the complex general linear group $\mathrm{GL}(N)$, where

$$J(n) = \left\{ h = \begin{pmatrix} h_0 & h_1 & \dots & h_{n-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & h_1 \\ & & & h_0 \end{pmatrix} \mid h_0, \dots, h_{n-1} \in \mathbb{C} \right\} \subset \mathrm{GL}(n)$$

and $(h^{(1)}, \dots, h^{(\ell)}) \in J(n_1) \times \cdots \times J(n_\ell)$ is identified with the block diagonal matrix $\mathrm{diag}(h^{(1)}, \dots, h^{(\ell)}) \in \mathrm{GL}(N)$. If we denote by Λ the shift matrix $(\delta_{i+1,j})_{1 \leq i,j \leq n}$ of size n, then $h \in J(n)$ is expressed as $h = \sum_{k=0}^{n-1} h_k \Lambda^k$. Gelfand's HGF of type λ is defined as a Radon transform of a character χ of the universal covering group $\tilde{H}_\lambda$. It is, roughly speaking in the case we are concerned, to substitute linear polynomials of $t = (t_0, t_1)$ into the character and integrate it on an appropriate path in $\mathbb{P}^1$ with the homogeneous coordinates t. A character χ of $\tilde{H}_\lambda$ means a Lie group homomorphism $\chi : \tilde{H}_\lambda \to \mathbb{C}^\times$ in this paper. To describe a character, we need the following. Let $x = (x_0, x_1, x_2, \cdots)$ be the variables and let T be the indeterminate. Define the functions $\theta_k(x) \quad (k \geq 0)$ by

$$\log(x_0 + x_1 T + x_2 T^2 + \cdots) = \log x_0 + \log \left(1 + \sum_{k=1}^{\infty} \frac{x_k}{x_0} T^k \right) = \sum_{k=0}^{\infty} \theta_k(x) T^k.$$

The right hand side is obtained by using $\log(1+X) = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} X^k$. Then $\theta_0 = \log x_0$ and

$$\theta_k(x) = \sum \frac{(-1)^{m+1}}{m} \left(\frac{x_{i_1}}{x_0} \right) \cdots \left(\frac{x_{i_m}}{x_0} \right) \quad (k \geq 1),$$

where the sum is taken over tuples $(i_1, \dots, i_m)$ with $i_1, \dots, i_m \geq 1$ and $i_1 + \cdots + i_m = k$. In particular, $\theta_1, \theta_2, \theta_3$ have the form

$$\theta_1(x) = \frac{x_1}{x_0}, \theta_2(x) = \frac{x_2}{x_0} - \frac{1}{2} \left(\frac{x_1}{x_0} \right)^2, \theta_3(x) = \frac{x_3}{x_0} - \left(\frac{x_1}{x_0} \right) \left(\frac{x_2}{x_0} \right) + \frac{1}{3} \left(\frac{x_1}{x_0} \right)^3.$$

Lemma 2.1. *[11] The correspondence*

$$h = \sum_{k=0}^{n-1} h_k \Lambda^k \mapsto (h_0, \theta_1(h), \dots, \theta_{n-1}(h))$$

gives a group isomorphism $J(n) \to \mathbb{C}^\times \times \mathbb{C}^{n-1}$, where $\mathbb{C}^{n-1}$ denotes the additive group with the vector addition.

By this lemma, we see that the universal covering group $\tilde{J}(n)$ of $J(n)$ is isomorphic to $\tilde{\mathbb{C}^\times} \times \mathbb{C}^{n-1}$. Then we can describe the characters of $\tilde{J}(n)$ and $\tilde{H}_\lambda$.

Lemma 2.2. *For a character χ of $\tilde{J}(n)$, there exist $\alpha = (\alpha_0, \dots, \alpha_{n-1}) \in \mathbb{C}^n$ such that*

$$\chi(h) = h_0^{\alpha_0} \exp\left(\sum_{k=1}^{n-1} \alpha_k \theta_k(h) \right), \quad h = \sum_{k=0}^{n-1} h_k \Lambda^k.$$

This character will be denoted as $\chi_n(\cdot, \alpha)$.

Lemma 2.3. *Any character $\chi : \tilde{H}_\lambda \to \mathbb{C}^\times$ is given by*

$$\chi(h, \alpha) = \prod_{i=1}^{\ell} \chi_{n_i}(h^{(i)}, \alpha^{(i)}), \quad h = \mathrm{diag}(h^{(1)}, \dots, h^{(\ell)}), \quad h^{(i)} \in \tilde{J}(n_i)$$

for some $\alpha = (\alpha^{(1)}, \dots, \alpha^{(\ell)}) \in \mathbb{C}^N, \quad \alpha^{(i)} = (\alpha_0^{(i)}, \alpha_1^{(i)}, \dots, \alpha_{n_i-1}^{(i)}) \in \mathbb{C}^{n_i}$.

For a character $\chi(\cdot, \alpha)$ of $\tilde{H}_\lambda$, we assume the condition

$$\alpha_{n_i-1}^{(i)} \begin{cases} \neq 0 & \text{if } n_i \geq 2, \\ \notin \mathbb{Z} & \text{if } n_i = 1, \end{cases} \qquad \alpha_0^{(1)} + \dots + \alpha_0^{(\ell)} = -2. \tag{2.5}$$

To consider the Radon transform of a character of $\tilde{H}_\lambda$, we prepare N linear polynomials of $t = (t_0, t_1)$ by specifying the coefficients of them by a matrix $z \in \mathrm{Mat}(2, N)$:

$$z = (z^{(1)}, \dots, z^{(\ell)}), \ z^{(i)} = (z_0^{(i)}, \dots, z_{n_i-1}^{(i)}), \ z_k^{(i)} = \begin{pmatrix} z_{0,k}^{(i)} \\ z_{1,k}^{(i)} \end{pmatrix} \in \mathbb{C}^2. \tag{2.6}$$

Let Z_λ be a Zariski open subset of $\mathrm{Mat}(2, N)$ defined by

$$Z_\lambda := \left\{ z \in \mathrm{Mat}(2, N) \mid \begin{array}{c} \det(z_0^{(i)}, z_0^{(j)}) \neq 0 \quad \text{for } i \neq j, \\ \det(z_0^{(i)}, z_1^{(i)}) \neq 0 \quad \text{for } i \text{ with } n_i \geq 2 \end{array} \right\}. \tag{2.7}$$

For $z \in Z_\lambda$, we define N linear polynomials by

$$tz = (tz^{(1)}, \dots, tz^{(l)}), \quad tz^{(i)} = (tz_0^{(i)}, \dots, tz_{n_i-1}^{(i)}), \quad tz_k^{(i)} = t_0 z_{0,k}^{(i)} + t_1 z_{1,k}^{(i)}.$$

We use the convention that $tz^{(i)}$ is identified with $\sum_{0 \leq k < n_i} (tz_k^{(i)}) \Lambda^k \in \tilde{J}(n_i)$ and tz is regarded as an element of $\tilde{H}_\lambda$.

Definition 2.4. The Gelfand's HGF of type λ is the function on Z_λ defined by

$$F(z,\alpha;C)=\int_C \chi(tz,\alpha)\cdot\tau, \quad \tau=t_0dt_1-t_1dt_0.$$

where C is an appropriate path in the t-space $\mathbb{P}^1$ representing an element of certain homology group.

Using the fact $\tau = t_0^2 d(t_1/t_0)$ and the condition (2.5) on α, we can write the above integral in terms of the affine coordinate $u=t_1/t_0$ in the affine chart $\{[t]\in\mathbb{P}^1 \mid t_0\neq 0\}$ of $\mathbb{P}^1$. Put $\vec{u}=(1,u)$. Then

$$F(z,\alpha;C)=\int_C \chi(\vec{u}z,\alpha)du.$$

The following property of Gelfand's HGF of type λ is important for the derivation of contiguity relations for the classical HGF family. Consider the action of $\mathrm{GL}(2)\times H_\lambda$ on $\mathrm{Mat}(2,N)$ defined by

$$\mathrm{GL}(2)\times\mathrm{Mat}(2,N)\times H_\lambda\ni(g,z,h)\mapsto gzh\in\mathrm{Mat}(2,N).$$

Then we see that it induces the action of $\mathrm{GL}(2)\times H_\lambda$ on Z_λ.

Proposition 2.5. *The following identities hold.*

(1) $F(zh,\alpha;C)=\chi(h,\alpha)F(z,\alpha;C), \qquad h\in H_\lambda,$

(2) $F(gz,\alpha;C)=(\det g)^{-1}F(z,\alpha;\tilde{C}), \qquad g\in\mathrm{GL}(2),$

where $\tilde{C}=\{\tilde{C}(z)\}$ *is obtained from* $C(z)$ *by the projective transformation* $\mathbb{P}^1\ni[t]\mapsto[s]:=[tg]\in\mathbb{P}^1$.

2.2 Contiguity relation for Gelfand's HGF

We recall how the contiguity relations for Gelfand's HGF are described. See also [10]. Let $\mathfrak{g}=\mathfrak{gl}(N)$ be the Lie algebra of $G=\mathrm{GL}(N)$ and $\mathfrak{h}_\lambda$ be the Lie algebra of H_λ which is an abelian Lie subalgebra of $\mathfrak{g}$. Since $H_\lambda=J(n_1)\times\cdots\times J(n_\ell)$, we see that $\mathfrak{h}_\lambda=\mathfrak{j}(n_1)\oplus\cdots\oplus\mathfrak{j}(n_\ell)$, where $\mathfrak{j}(n)$ denotes the Lie algebra of $J(n)$. Let $X\in\mathfrak{h}_\lambda$ be expressed as

$$X=\mathrm{diag}(X^{(1)},\dots,X^{(\ell)})\in\mathrm{Mat}(N), \quad X^{(i)}=\sum_{0\le k<n_i}X_k^{(i)}\Lambda^k\in\mathfrak{j}(n_i) \tag{2.8}$$

with $X_k^{(i)}\in\mathbb{C}$ and the shift matrix Λ. Since $\mathfrak{h}_\lambda$ is abelian, we have a commuting family of Lie algebra homomorphisms $\{\mathrm{ad}_X\in \mathit{End}(\mathfrak{g})\mid X\in\mathfrak{h}_\lambda\}$, where $\mathrm{ad}_X(Y):=[X,Y]:=XY-YX$. Then we can decompose $\mathfrak{g}$ into the simultaneous generalized eigenspaces with respect to this commuting family of endomorphisms:

$$\mathfrak{g}=\mathfrak{h}_\lambda\oplus\bigoplus_{\alpha\in\Delta}\mathfrak{g}_\alpha, \quad \Delta=\{e^{(i)}-e^{(j)}\mid 1\le i\neq j\le\ell\}. \tag{2.9}$$

Here, $e^{(i)}$ is the element of the dual space $\mathfrak{h}_\lambda^*$ of $\mathfrak{h}_\lambda$ which maps an element X of the form (2.8) to $X_0^{(i)} \in \mathbb{C}$, Δ denotes the set of roots and

$$\mathfrak{g}_{e^{(i)}-e^{(j)}} = \left\{ Y \in \mathfrak{g} \mid \left(\mathrm{ad}_X - (e^{(i)} - e^{(j)})(X) \right)^m Y = 0 \text{ for } \forall X \in \mathfrak{h}_\lambda, \exists m \in \mathbb{N} \right\}.$$

We can see that each generalized eigenspace $\mathfrak{g}_{e^{(i)}-e^{(j)}}$ contains 1-dimensional eigenspace with a basis vector $E_{\epsilon^{(i)}-\epsilon^{(j)}}$ which is the matrix unit given as follows. According as the partition $\lambda = (n_1, \ldots, n_\ell)$ we express $Y \in \mathfrak{g}$ in block-wise as

$$Y = \begin{pmatrix} Y^{(1,1)} & \ldots & Y^{(1,\ell)} \\ \vdots & & \vdots \\ Y^{(\ell,1)} & \ldots & Y^{(\ell,\ell)} \end{pmatrix}$$

with the (i,j) block $Y^{(i,j)} \in \mathrm{Mat}(n_i, n_j)$. Then $E_{\epsilon^{(i)}-\epsilon^{(j)}}$ is the matrix unit whose only nonzero element 1 locates at the upper right corner of (i,j) block. The contiguity relations for Gelfand's HGF of type λ are provided by the infinitesimal action of the 1-paramter subgroup $s \mapsto \exp(sE_{\epsilon^{(i)}-\epsilon^{(j)}})$. Let Y be a root vector in the generalized root space decomposition (2.9). Then we see that the 1-parameter subgroup $\{\exp(sY)\}_{s\in\mathbb{C}}$ preserves the space Z_λ and acts on functions on it. Then we have its infinitesimal action in the form of the first order differential operators on Z_λ. Namely, for a holomorphic function f on Z_λ, define the differential operator L_Y by

$$(L_Y f)(z) := \frac{d}{ds} f(z \exp(sY))|_{s=0}. \tag{2.10}$$

When $Y = E_{\epsilon^{(i)}-\epsilon^{(j)}}$, the operator L_Y will be denoted by $L_{\epsilon^{(i)}-\epsilon^{(j)}}$.

Proposition 2.6. *Let F be Gelfand's HGF of type λ. Then we have*

$$L_{\epsilon^{(i)}-\epsilon^{(j)}} F(z, \alpha) = \alpha^{(j)}_{n_j-1} F(z, \alpha + \epsilon^{(i)} - \epsilon^{(j)}), \tag{2.11}$$

where $\alpha \mapsto \alpha + \epsilon^{(i)} - \epsilon^{(j)}$ implies, for $\alpha = (\alpha^{(1)}, \ldots, \alpha^{(\ell)}), \alpha^{(k)} = (\alpha_0^{(k)}, \ldots, \alpha^{(k)}_{n_k-1})$, the change $\alpha_0^{(i)} \mapsto \alpha_0^{(i)} + 1, \alpha_0^{(j)} \mapsto \alpha_0^{(j)} - 1$ leaving the other entries unchanged.

2.3 Classical HGFs as Gelfand's HGF

Classical HGFs are realized as Gelfand's HGF considering the case $N = 4$ and partitions of 4 given in (2.4). For each partition λ in (2.4), there corresponds Gauss, Kummer, Bessel, Hermite-Weber and Airy function, respectively, which we shall explain in the following. For a λ of 4, the character of $\tilde{H}_\lambda$ and the space Z_λ, on which Gelfand's HGF of type λ is defined, are as follows. Here we use a slightly different manner of the indices from that of the above subsections in order to avoid the cumbersome notation.

Group H_λ:

$$H_{(1,1,1,1)} = \left\{ \begin{pmatrix} h_1 & & & \\ & h_2 & & \\ & & h_3 & \\ & & & h_4 \end{pmatrix} \right\}, \quad H_{(3,1)} = \left\{ \begin{pmatrix} h_1 & h_2 & h_3 & \\ & h_1 & h_2 & \\ & & h_1 & \\ & & & h_4 \end{pmatrix} \right\},$$

$$H_{(2,1,1)} = \left\{ \begin{pmatrix} h_1 & h_2 & & \\ & h_1 & & \\ & & h_3 & \\ & & & h_4 \end{pmatrix} \right\}, \quad H_{(4)} = \left\{ \begin{pmatrix} h_1 & h_2 & h_3 & h_4 \\ & h_1 & h_2 & h_3 \\ & & h_1 & h_2 \\ & & & h_1 \end{pmatrix} \right\},$$

$$H_{(2,2)} = \left\{ \begin{pmatrix} h_1 & h_2 & & \\ & h_1 & & \\ & & h_3 & h_4 \\ & & & h_3 \end{pmatrix} \right\}.$$

Character χ_λ:

$$\chi_{(1,1,1,1)}(h,\alpha) = h_1^{\alpha_1} h_2^{\alpha_2} h_3^{\alpha_3} h_4^{\alpha_4},$$

$$\chi_{(2,1,1)}(h,\alpha) = h_1^{\alpha_1} \exp\left(\alpha_2 \frac{h_2}{h_1}\right) h_3^{\alpha_3} h_4^{\alpha_4},$$

$$\chi_{(2,2)}(h,\alpha) = h_1^{\alpha_1} \exp\left(\alpha_2 \frac{h_2}{h_1}\right) h_3^{\alpha_3} \exp\left(\alpha_4 \frac{h_4}{h_3}\right),$$

$$\chi_{(3,1)}(h,\alpha) = h_1^{\alpha_1} \exp\left(\alpha_2 \frac{h_2}{h_1} + \alpha_3 \left(\frac{h_3}{h_1} - \frac{1}{2}\left(\frac{h_2}{h_1}\right)^2\right)\right) h_4^{\alpha_4},$$

$$\chi_{(4)}(h,\alpha) = h_1^{\alpha_1} \exp\left\{ \alpha_2 \frac{h_2}{h_1} + \alpha_3 \left(\frac{h_3}{h_1} - \frac{1}{2}\left(\frac{h_2}{h_1}\right)^2\right) + \alpha_4 \left(\frac{h_4}{h_1} - \frac{h_2}{h_1}\frac{h_3}{h_1} + \frac{1}{3}\left(\frac{h_2}{h_1}\right)^3\right) \right\}.$$

Matrix space Z_λ:

$$Z_{(1,1,1,1)} = \{(z_1,z_2,z_3,z_4) \in \mathrm{Mat}(2,4) \mid \det(z_i,z_j) \neq 0 \ (i \neq j)\},$$

$$Z_{(2,1,1)} = \left\{ (z_1,z_2,z_3,z_4) \in \mathrm{Mat}(2,4) \;\middle|\; \begin{array}{l} \det(z_1,z_j) \neq 0 \ (2 \leq j \leq 4) \\ \det(z_3,z_4) \neq 0 \end{array} \right\},$$

$$Z_{(2,2)} = \left\{ (z_1,z_2,z_3,z_4) \in \mathrm{Mat}(2,4) \;\middle|\; \begin{array}{l} \det(z_1,z_j) \neq 0 \ (2 \leq j \leq 3) \\ \det(z_3,z_4) \neq 0 \end{array} \right\},$$

$$Z_{(3,1)} = \left\{ (z_1,z_2,z_3,z_4) \in \mathrm{Mat}(2,4) \;\middle|\; \begin{array}{l} \det(z_1,z_2) \neq 0 \\ \det(z_1,z_4) \neq 0 \end{array} \right\},$$

$$Z_{(4)} = \{(z_1,z_2,z_3,z_4) \in \mathrm{Mat}(2,4) \mid \det(z_1,z_2) \neq 0\}.$$

Taking into account Proposition 2.5, consider the orbit space $\mathrm{GL}(2)\backslash Z_\lambda / H_\lambda$ of the action $\mathrm{GL}(2) \curvearrowright Z_\lambda \curvearrowleft H_\lambda$. Then we can have the realization X_λ of the orbit space in Z_λ as follows.

$$X_{(1,1,1,1)} = \left\{ \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -x \end{pmatrix} \mid x \neq 0,1 \right\}, \quad X_{(3,1)} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & x & 1 \end{pmatrix} \right\},$$
$$X_{(2,1,1)} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & x & 1 & -1 \end{pmatrix} \mid x \neq 0 \right\}, \quad X_{(4)} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -x \end{pmatrix} \right\},$$
$$X_{(2,2)} = \left\{ \begin{pmatrix} 1 & 0 & 0 & -x \\ 0 & 1 & 1 & 0 \end{pmatrix} \mid x \neq 0 \right\}.$$

Then the classical HGF family can be identified as Gelfand's HGF on X_λ with the parameters chosen appropriately.

(1) $\lambda = (1,1,1,1) \leftrightarrow$ Gauss:

$$\begin{aligned} \alpha &= (\alpha_1, \alpha_2, \alpha_3, \alpha_4) := (b-c, a-1, c-a-1, -b), \\ \mathbf{x} &= (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -x \end{pmatrix}, \\ F(\mathbf{x}, \alpha; C) &= \int_C (\vec{u}\mathbf{x}_1)^{\alpha_1} (\vec{u}\mathbf{x}_2)^{\alpha_2} (\vec{u}\mathbf{x}_3)^{\alpha_3} (\vec{u}\mathbf{x}_4)^{\alpha_4} du \\ &= \int_C 1^{\alpha_1} u^{\alpha_2} (1-u)^{\alpha_3} (1-ux)^4 du. \end{aligned} \tag{2.12}$$

(2) $\lambda = (2,1,1) \leftrightarrow$ Kummer:

$$\begin{aligned} \alpha &= (\alpha_1, \alpha_2, \alpha_3, \alpha_4) := (-c, 1, a-1, c-a-1), \\ \mathbf{x} &= (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & x & 1 & -1 \end{pmatrix}, \\ F(\mathbf{x}, \alpha) &= \int_C (\vec{u}\mathbf{x}_1)^{\alpha_1} \exp\left(\alpha_2 \frac{\vec{u}\mathbf{x}_2}{\vec{u}\mathbf{x}_1} \right) (\vec{u}\mathbf{x}_3)^{\alpha_3} (\vec{u}\mathbf{x}_4)^{\alpha_4} du \\ &= \int_C 1^{\alpha_1} \exp(xu) u^{\alpha_3} (1-u)^{\alpha_4} du. \end{aligned} \tag{2.13}$$

(3) $\lambda = (2,2) \leftrightarrow$ Bessel:

$$\begin{aligned} \alpha &= (\alpha_1, \alpha_2, \alpha_3, \alpha_4) := (c-1, 1, -c-1, 1), \\ \mathbf{x} &= (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \begin{pmatrix} 1 & 0 & 0 & -x \\ 0 & 1 & 1 & 0 \end{pmatrix}, \\ F(\mathbf{x}, \alpha) &= \int_C (\vec{u}\mathbf{x}_1)^{\alpha_1} \exp\left(\frac{\vec{u}\mathbf{x}_2}{\vec{u}\mathbf{x}_1} \right) (\vec{u}\mathbf{x}_3)^{\alpha_3} \exp\left(\frac{\vec{u}\mathbf{x}_4}{\vec{u}\mathbf{x}_3} \right) du \\ &= \int_C 1^{\alpha_1} \exp(u) u^{\alpha_3} \exp(-x/u) du. \end{aligned} \tag{2.14}$$

(4) $\lambda = (3,1) \leftrightarrow$ Hermite-Weber:

$$\begin{aligned}
\alpha &= (\alpha_1,\alpha_2,\alpha_3,\alpha_4) := (a-1,0,1,-a-1),\\
\mathbf{x} &= (\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & x & 1\end{pmatrix},\\
F(\mathbf{x},\alpha) &= \int_C (\vec{u}\mathbf{x}_1)^{\alpha_1} \exp\left\{\frac{\vec{u}\mathbf{x}_3}{\vec{u}\mathbf{x}_1} - \frac{1}{2}\left(\frac{\vec{u}\mathbf{x}_2}{\vec{u}\mathbf{x}_1}\right)^2\right\}(\vec{u}\mathbf{x}_4)^{\alpha_4} du \qquad (2.15)\\
&= \int_C 1^{\alpha_1} \exp\left(xu - \frac{1}{2}u^2\right) u^{\alpha_4} du.
\end{aligned}$$

(5) $\lambda = (4) \leftrightarrow$ Airy:

$$\begin{aligned}
\alpha &= (\alpha_1,\alpha_2,\alpha_3,\alpha_4) := (-2,0,0,-1),\\
\mathbf{x} &= (\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & -x\end{pmatrix},\\
F(\mathbf{x},\alpha) &= \int_C (\vec{u}\mathbf{x}_1)^{\alpha_1} \exp\alpha_4\left\{\frac{\vec{u}\mathbf{x}_4}{\vec{u}\mathbf{x}_1} - \frac{\vec{u}\mathbf{x}_2}{\vec{u}\mathbf{x}_1}\frac{\vec{u}\mathbf{x}_3}{\vec{u}\mathbf{x}_1} + \frac{1}{3}\left(\frac{\vec{u}\mathbf{x}_2}{\vec{u}\mathbf{x}_1}\right)^3\right\} du\\
&= \int_C 1^{\alpha_1} \exp\left(xu - \frac{1}{3}u^3\right) du.
\end{aligned}$$

3 Contiguity relations of the classical HGF

In this section, we derive the explicit form of contiguity relations for the classical HGF from Proposition 2.6.

3.1 Gauss case

As we have seen in Subsection 2.3, the Gauss HGF ${}_2F_1(a,b,c;x)$ is identified with Gelfand's HGF of type $\lambda = (1,1,1,1)$ on the realization $X = X_\lambda \subset Z_\lambda$ of the quotient space $\mathrm{GL}(2)\backslash Z_\lambda/H_\lambda$. We adopt the notation used in Subsection 2.3 for numbering the indices for matrices, etc. In accordance with this convention of notations, the rows and columns for $Y \in \mathfrak{gl}(4)$ are indexed by $\{1,2,3,4\}$. Then Proposition 2.6 gives the following result.

Proposition 3.1. *The contiguity relations for Gauss HGF ${}_2F_1(a,b,c;x)$ are given as follows according to the roots $\Delta = \{e^{(i)} - e^{(j)} \mid 1 \le i \ne j \le 4\}$.*

(1) $\pm(e^{(1)} - e^{(2)})$:

$$\begin{aligned}
L_{e^{(1)}-e^{(2)}}\, {}_2F_1(a,b,c;x) &= (c-1)\cdot {}_2F_1(a-1,b,c-1;x),\\
L_{e^{(2)}-e^{(1)}}\, {}_2F_1(a,b,c;x) &= \frac{a(b-c)}{c}\cdot {}_2F_1(a+1,b,c+1;x),
\end{aligned}$$

where

$$L_{e^{(1)}-e^{(2)}} = x(1-x)\frac{d}{dx} + c - 1 - bx,$$
$$L_{e^{(2)}-e^{(1)}} = (1-x)\frac{d}{dx} - a.$$

(2) $\pm(e^{(1)} - e^{(3)}):$

$$L_{e^{(1)}-e^{(3)}}\ {}_2F_1(a,b,c;x) = (c-1)\cdot\ {}_2F_1(a,b,c-1;x),$$
$$L_{e^{(3)}-e^{(1)}}\ {}_2F_1(a,b,c;x) = \frac{(c-a)(c-b)}{c}\cdot\ {}_2F_1(a,b,c+1;x),$$

where

$$L_{e^{(1)}-e^{(3)}} = x\frac{d}{dx} + c - 1,$$
$$L_{e^{(3)}-e^{(1)}} = -(1-x)\frac{d}{dx} + a + b - c.$$

(3) $\pm(e^{(1)} - e^{(4)}):$

$$L_{e^{(1)}-e^{(4)}}\ {}_2F_1(a,b,c;x) = (-b)\cdot\ {}_2F_1(a,b+1,c;x),$$
$$L_{e^{(4)}-e^{(1)}}\ {}_2F_1(a,b,c;x) = -(c-b)\cdot\ {}_2F_1(a,b-1,c;x),$$

where

$$L_{e^{(1)}-e^{(4)}} = -x\frac{d}{dx} - b,$$
$$L_{e^{(4)}-e^{(1)}} = -x(1-x)\frac{d}{dx} + b - c + ax.$$

(4) $\pm(e^{(2)} - e^{(3)}):$

$$L_{e^{(2)}-e^{(3)}}\ {}_2F_1(a,b,c;x) = a\cdot\ {}_2F_1(a+1,b,c;x),$$
$$L_{e^{(3)}-e^{(2)}}\ {}_2F_1(a,b,c;x) = (c-a)\cdot\ {}_2F_1(a-1,b,c;x),$$

where

$$L_{e^{(2)}-e^{(3)}} = x\frac{d}{dx} + a,$$
$$L_{e^{(3)}-e^{(2)}} = x(1-x)\frac{d}{dx} + c - a - bx.$$

(5) $\pm(e^{(2)} - e^{(4)}):$

$$L_{e^{(2)}-e^{(4)}}\ {}_2F_1(a,b,c;x) = -\frac{ab}{c}\cdot\ {}_2F_1(a+1,b+1,c+1;x),$$
$$L_{e^{(4)}-e^{(2)}}\ {}_2F_1(a,b,c;x) = (c-1)\cdot\ {}_2F_1(a-1,b-1,c-1;x),$$

where

$$L_{e^{(2)}-e^{(4)}} = -\frac{d}{dx},$$
$$L_{e^{(4)}-e^{(2)}} = x(1-x)\frac{d}{dx} + c - 1 + (1-a-b)x.$$

(6) $\pm(e^{(3)} - e^{(4)})$:

$$L_{e^{(3)}-e^{(4)}}\, {}_2F_1(a,b,c;x) = -\frac{b(c-a)}{c} \cdot {}_2F_1(a,b+1,c+1;x),$$
$$L_{e^{(4)}-e^{(3)}}\, {}_2F_1(a,b,c;x) = (c-1) \cdot {}_2F_1(a,b-1,c-1;x),$$

where

$$L_{e^{(3)}-e^{(4)}} = (1-x)\frac{d}{dx} - b,$$
$$L_{e^{(4)}-e^{(3)}} = x(1-x)\frac{d}{dx} + c - 1 - ax.$$

Proof. We show only the case (1) since the other cases are shown in a similar way. Take

$$\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -x \end{pmatrix} \in X.$$

Then, comparing (2.2) and (2.12), the Gauss HGF and Gelfand's HGF restricted on X are related as

$$F(\mathbf{x}, \alpha) = C(\alpha)\, {}_2F_1(\alpha_2 + 1, -\alpha_4, \alpha_2 + \alpha_3 + 2; x), \tag{3.1}$$

where the path of integration in the left hand side is $\overrightarrow{0,1}$ connecting 0 to 1 in u-plane, the parameters α and those of Gauss HGF are related as

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (b - c, a - 1, c - a - 1, -b) \tag{3.2}$$

or

$$(a, b, c) = (\alpha_2 + 1, -\alpha_4, \alpha_2 + \alpha_3 + 2)$$

and the constant $C(\alpha)$ is

$$C(\alpha) = \frac{\Gamma(\alpha_2 + 1)\Gamma(\alpha_3 + 1)}{\Gamma(\alpha_2 + \alpha_3 + 2)} = \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}. \tag{3.3}$$

To obtain the contiguity relations, consider the action of the 1-parameter subgroup generated by the matrix unit $E_{1,2}$(resp. $E_{2,1}$), which is a root vector corresponding to the root $e^{(1)} - e^{(2)}$ (resp. $e^{(2)} - e^{(1)}$) in the root space decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{gl}(4)$. Then Proposition 2.6 says for the root $e^{(1)} - e^{(2)}$ that

$$L_{e^{(1)}-e^{(2)}} F(\mathbf{x}, \alpha) = \alpha_2 F(\mathbf{x}, \alpha + e^{(1)} - e^{(2)}) \tag{3.4}$$

with the operator defined by

$$(L_{e^{(1)}-e^{(2)}}\cdot f)(\mathbf{x}) = \frac{d}{ds} f(\mathbf{x}\exp(sE_{1,2}))|_{s=0}. \tag{3.5}$$

Let us compute the operator $L_{e^{(1)}-e^{(2)}}$. Consider the 1-parameter subgroup generated by $E_{1,2}$ and its action on $\mathbf{x}$:

$$\begin{aligned} z(s) = \mathbf{x}\exp(sE_{1,2}) &= (\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)\begin{pmatrix} 1 & s & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \\ &= (\mathbf{x}_1,\mathbf{x}_2+s\mathbf{x}_1,\mathbf{x}_3,\mathbf{x}_4) = \begin{pmatrix} 1 & s & 1 & 1 \\ 0 & 1 & -1 & -x \end{pmatrix}. \end{aligned}$$

We normalize $z(s)$ to the normal form $\mathbf{x}(s)\in X$. Put $g_1 = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ and $h = \mathrm{diag}(1,h_2,h_3,h_4)\in H$ and consider

$$g_1^{-1}z(s)h = \begin{pmatrix} 1 & 0 & (1+s)h_3 & (1+sx)h_4 \\ 0 & h_2 & -h_3 & -xh_4 \end{pmatrix}.$$

Next take $g_2 = \begin{pmatrix} 1 & \\ & h_2 \end{pmatrix}$ and consider

$$g_2^{-1}g_1^{-1}z(s)h = \begin{pmatrix} 1 & 0 & (1+s)h_3 & (1+sx)h_4 \\ 0 & 1 & -h_2^{-1}h_3 & -xh_2^{-1}h_4 \end{pmatrix}.$$

Determine h as $h_2 = h_3 = (1+s)^{-1}, h_4 = (1+sx)^{-1}$ so that we have

$$g_2^{-1}g_1^{-1}z(s)h = \mathbf{x}(s) = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -x(s) \end{pmatrix}, \quad x(s) = \frac{(1+s)x}{1+sx}.$$

It follows from Proposition 2.5 that

$$\begin{aligned} F(z(s),\alpha) &= \det(g_1g_2)^{-1}\chi(h,\alpha)^{-1}F(\mathbf{x}(s),\alpha) \\ &= (1+s)^{\alpha_2+\alpha_3+1}(1+sx)^{\alpha_4}F(\mathbf{x}(s),\alpha). \end{aligned} \tag{3.6}$$

Then the left hand side of (3.4) is computed as

$$\begin{aligned} L_{e^{(1)}-e^{(2)}}F(\mathbf{x},\alpha) &= \frac{d}{ds}F(z(s),\alpha)|_{s=0} \\ &= \frac{d}{ds}(1+s)^{\alpha_2+\alpha_3+1}(1+sx)^{\alpha_4}F(\mathbf{x}(s),\alpha)|_{s=0} \\ &= (\alpha_2+\alpha_3+1+\alpha_4 x)F(\mathbf{x},\alpha) + \frac{\partial F(\mathbf{x},\alpha)}{\partial x}\frac{dx(s)}{ds}|_{s=0} \\ &= \left\{ x(1-x)\frac{d}{dx} + \alpha_2+\alpha_3+1+\alpha_4 x \right\} F(\mathbf{x},\alpha). \end{aligned}$$

Hence the operator $L_{e^{(1)}-e^{(2)}}$ defined by (3.5) is given by

$$L_{e^{(1)}-e^{(2)}} = x(1-x)\frac{d}{dx} + \alpha_2 + \alpha_3 + 1 + \alpha_4 x.$$

Taking account (3.2) and

$$\begin{aligned}\frac{C(\alpha + e^{(1)} - e^{(2)})}{C(\alpha)} &= \frac{\Gamma(\alpha_2+\alpha_3+2)}{\Gamma(\alpha_2+1)\Gamma(\alpha_3+1)} \cdot \frac{\Gamma(\alpha_2)\Gamma(\alpha_3+1)}{\Gamma(\alpha_2+\alpha_3+1)} \\ &= \frac{\alpha_2+\alpha_3+1}{\alpha_2},\end{aligned}$$

the relation (3.4) is translated to the first half of the contiguity relations in (1) for the Gauss HGF:

$$\left\{x(1-x)\frac{d}{dx} + c - 1 - bx\right\} {}_2F_1(a,b,c;x) = (c-1)\cdot {}_2F_1(a-1,b,c-1;x).$$

In a similar way, we shall compute the second half of the contiguity relation corresponding to the root $e^{(2)} - e^{(1)}$. Consider the 1-parameter subgroup generated by the corresponding root vector $E_{2,1}$ and its action on X:

$$z(s) = \mathbf{x}\exp(sE_{2,1}) = \begin{pmatrix} 1 & 0 & 1 & 1 \\ s & 1 & -1 & -x \end{pmatrix}.$$

We normalize $z(s)$ to the normal form. Take $g_1 = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$, $h = \mathrm{diag}(h_1, 1, h_3, h_4)$ and get

$$g_1^{-1} z(s) h = \begin{pmatrix} h_1 & 0 & h_3 & h_4 \\ 0 & 1 & -(1+s)h_3 & -(x+s)h_4 \end{pmatrix}.$$

Next take $g_2 = \begin{pmatrix} h_1 & \\ & 1 \end{pmatrix}$ and consider

$$g_2^{-1} g_1^{-1} z(s) h = \begin{pmatrix} 1 & 0 & h_1^{-1}h_3 & h_1^{-1}h_4 \\ 0 & 1 & -(1+s)h_3 & -(x+s)h_4 \end{pmatrix}.$$

Determining h by $h_1 = h_3 = h_4 = (1+s)^{-1}$, we have

$$g_2^{-1} g_1^{-1} z(s) h = \mathbf{x}(s)$$

with

$$\mathbf{x}(s) = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -x(s) \end{pmatrix}, \qquad x(s) = \frac{x+s}{1+s}.$$

Then from Proposition 2.5, we have

$$\begin{aligned} F(z(s),\alpha) &= \det(g_1 g_2)^{-1}\chi(h,\alpha)^{-1} F(\mathbf{x}(s),\alpha) \\ &= (1+s)^{\alpha_1+\alpha_3+\alpha_4+1} F(\mathbf{x}(s),\alpha). \end{aligned} \tag{3.7}$$

Let us compute the left hand side of (3.4) using (3.7). Noting $\alpha_1+\alpha_3+\alpha_4+1 = -\alpha_2-1$, we have

$$\begin{aligned} L_{e^{(2)}-e^{(1)}}F(\mathbf{x},\alpha) &= \frac{d}{ds}F(z(s),\alpha)|_{s=0} \\ &= \frac{d}{ds}(1+s)^{-\alpha_2-1}F(\mathbf{x}(s),\alpha)|_{s=0} \\ &= (-\alpha_2-1)F(\mathbf{x},\alpha)+\frac{\partial F(\mathbf{x},\alpha)}{\partial x}\frac{dx(s)}{ds} \\ &= \left\{(1-x)\frac{d}{dx}-\alpha_2-1\right\}F(\mathbf{x},\alpha). \end{aligned}$$

So (3.4) implies

$$\left\{(1-x)\frac{d}{dx}-\alpha_2-1\right\}F(\mathbf{x},\alpha)=\alpha_1F(\mathbf{x},\alpha+e^{(2)}-e^{(1)}).$$

Taking account of (3.2) and

$$\begin{aligned} \frac{C(\alpha+e^{(2)}-e^{(1)})}{C(\alpha)} &= \frac{\Gamma(\alpha_2+\alpha_3+2)}{\Gamma(\alpha_2+1)\Gamma(\alpha_3+1)}\cdot\frac{\Gamma(\alpha_2+2)\Gamma(\alpha_3+1)}{\Gamma(\alpha_2+\alpha_3+3)} \\ &= \frac{\alpha_2+1}{\alpha_2+\alpha_3+2}, \end{aligned}$$

the relation (3.4) is translated to the second half of the contiguity relations in (1) :

$$\left\{(1-x)\frac{d}{dx}-a\right\}{}_2F_1(a,b,c;x)=\frac{a(b-c)}{c}\cdot{}_2F_1(a+1,b,c+1;x).$$

Remark 3.2. In the above proposition, we showed that the contiguity operators for the Gauss HGF are obtained from the action of 1-parameter subgroup generated by root vectors $E_{e^{(i)}-e^{(j)}}$. It is to be noted that the operators $L_{e^{(i)}-e^{(j)}}$ does not necessarily satisfies the similar commutation relations for the root vectors $E_{e^{(i)}-e^{(j)}}$. For example, we know $[E_{e^{(1)}-e^{(2)}},E_{e^{(2)}-e^{(3)}}]=E_{e^{(1)}-e^{(3)}}$ holds for the root vectors, but we do not have $[L_{e^{(1)}-e^{(2)}},L_{e^{(2)}-e^{(3)}}]=L_{e^{(1)}-e^{(3)}}$. It come from the fact that we used Proposition 2.5 in order to reduce $z(s)=\mathbf{x}\exp(sE_{e^{(i)}-e^{(j)}})$ to the normal form $\mathbf{x}(s)$ and as a result the operator is twisted. The same remark should be added to the confluent family case.

3.2 Kummer case

From (2.3) and (2.13), Kummer's confluent HGF ${}_1F_1(a,c;x)$ is identified with Gelfand's HGF of type $\lambda=(2,1,1)$ on the realization $X=X_\lambda\subset Z_\lambda$ of the quotient space $\mathrm{GL}(2)\backslash Z_\lambda/H_\lambda$. We adopt the notation in Subsection 2.3 for numbering the entries of matrices. We have

$$F(\mathbf{x},\alpha)=C(\alpha)\,{}_1F_1(\alpha_3+1,\alpha_3+\alpha_4+2;x), \tag{3.8}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (-c, 1, a-1, c-a-1)$, the path of integration C in (2.13) is $\overrightarrow{0,1}$ and

$$C(\alpha) = \frac{\Gamma(\alpha_3+1)\Gamma(\alpha_4+1)}{\Gamma(\alpha_3+\alpha_4+2)} = \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}. \tag{3.9}$$

Then Proposition 2.6 gives the following result.

Proposition 3.3. *The contiguity relations for Kummer's confluent HGF* ${}_1F_1(a,c;x)$ *are given as follows according to the roots* $\Delta = \{e^{(i)} - e^{(j)} \mid 1 \le i \ne j \le 3\}$.

(1) $\pm(e^{(1)} - e^{(2)})$:

$$L_{e^{(1)}-e^{(2)}}\, {}_1F_1(a,c;x) = (c-1)\cdot {}_1F_1(a-1,c-1;x),$$
$$L_{e^{(2)}-e^{(1)}}\, {}_1F_1(a,c;x) = \frac{a}{c}\cdot {}_1F_1(a+1,c+1;x),$$

where

$$L_{e^{(1)}-e^{(2)}} = x\frac{d}{dx} + c - 1 - x, \quad L_{e^{(2)}-e^{(1)}} = \frac{d}{dx}.$$

(2) $\pm(e^{(1)} - e^{(3)})$:

$$L_{e^{(1)}-e^{(3)}}\, {}_1F_1(a,c;x) = (c-1)\cdot {}_1F_1(a,c-1;x),$$
$$L_{e^{(3)}-e^{(1)}}\, {}_1F_1(a,c;x) = \frac{c-a}{c}\cdot {}_1F_1(a,c+1;x),$$

where

$$L_{e^{(1)}-e^{(3)}} = x\frac{d}{dx} + c - 1, \quad L_{e^{(3)}-e^{(1)}} = -\frac{d}{dx} + 1.$$

(3) $\pm(e^{(2)} - e^{(3)})$:

$$L_{e^{(2)}-e^{(3)}}\, {}_1F_1(a,c;x) = a\cdot {}_1F_1(a+1,c;x),$$
$$L_{e^{(3)}-e^{(2)}}\, {}_1F_1(a,c;x) = (c-a)\cdot {}_1F_1(a-1,c;x),$$

where

$$L_{e^{(2)}-e^{(3)}} = x\frac{d}{dx} + a, \quad L_{e^{(3)}-e^{(2)}} = x\frac{d}{dx} + c - a - x.$$

Proof. We prove the case (1) only. Note that $\mathbf{x} \in X_{(2,1,1)}$ in (3.8) is given by $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & x & 1 & -1 \end{pmatrix}$. For the root $e^{(1)} - e^{(2)}$, we take the root vector given by the matrix unit $E_{1,3}$ and the action of the 1-parameter subgroup $\exp(sE_{1,3})$ on $\mathbf{x}$. Then we have

$$z(s) = \mathbf{x}\exp(sE_{1,3}) = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)\begin{pmatrix} 1 & & s & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$
$$= (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 + s\mathbf{x}_1, \mathbf{x}_4) = \begin{pmatrix} 1 & 0 & s & 1 \\ 0 & x & 1 & -1 \end{pmatrix}.$$

Let us normalize $z(s)$ to the normal form $\mathbf{x}(s) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & x(s) & 1 & -1 \end{pmatrix}$ by the action of $\mathrm{GL}(2) \times H_{(2,1,1)}$. Take

$$g_1 = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & h_2 & & \\ & 1 & & \\ & & h_3 & \\ & & & h_4 \end{pmatrix}$$

and consider $g_1^{-1}z(s)h$. Then we have

$$g_1^{-1}z(s)h = \begin{pmatrix} 1 & h_2 - sx & 0 & (1+s)h_4 \\ 0 & x & h_3 & -h_4 \end{pmatrix}.$$

To normalize the first and third column vectors, take $g_2 = \begin{pmatrix} 1 & \\ & h_3 \end{pmatrix}$ and consider

$$g_2^{-1}g_1^{-1}z(s)h = \begin{pmatrix} 1 & h_2 - sx & 0 & (1+s)h_4 \\ 0 & h_3^{-1}x & 1 & -h_3^{-1}h_4 \end{pmatrix}.$$

Then taking h with $h_2 = sx, h_3 = h_4 = (1+s)^{-1}$, we have

$$g_2^{-1}g_1^{-1}z(s)h = \mathbf{x}(s) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & x(s) & 1 & -1 \end{pmatrix}, \quad x(s) = (1+s)x.$$

Then from Proposition 2.5 we have

$$\begin{aligned} F(z(s),\alpha) &= \det(g_1g_2)^{-1}\chi(h,\alpha)^{-1}F(\mathbf{x}(s),\alpha) \\ &= e^{-sx}(1+s)^{\alpha_3+\alpha_4+1}F(\mathbf{x}(s),\alpha). \end{aligned} \tag{3.10}$$

We compute the left hand side of (2.11) for the root $e^{(1)} - e^{(2)}$ using (3.10).

$$\begin{aligned} L_{e^{(1)}-e^{(2)}}F(\mathbf{x},\alpha) &= \frac{d}{ds}F(z(s),\alpha)|_{s=0} \\ &= \frac{d}{ds}e^{-sx}(1+s)^{\alpha_3+\alpha_4+1}F(\mathbf{x}(s),\alpha)|_{s=0} \\ &= \left\{x\frac{d}{dx} + \alpha_3 + \alpha_4 + 1 - x\right\}F(\mathbf{x},\alpha). \end{aligned}$$

So (2.11) implies

$$\left\{x\frac{d}{dx} + \alpha_3 + \alpha_4 + 1 - x\right\}F(\mathbf{x},\alpha) = \alpha_3 F(\mathbf{x},\alpha + e^{(1)} - e^{(2)}).$$

Noting $(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = (-c, 1, a-1, c-a-1), \alpha + e^{(1)} - e^{(2)} = (\alpha_1 + 1, \alpha_2, \alpha_3 - 1, \alpha_4)$ and (3.9), which implies

$$\frac{C(\alpha + e^{(1)} - e^{(2)})}{C(\alpha)} = \frac{\alpha_3 + \alpha_4 + 1}{\alpha_3},$$

we have the contiguity relation

$$\left\{x\frac{d}{dx}+c-1-x\right\}\,{}_1F_1(a,c;x)=(c-1)\cdot\,{}_1F_1(a-1,c-1;x)$$

with the contiguity operator $L_{e^{(1)}-e^{(2)}}=x\frac{d}{dx}+c-1-x$.

Next we consider the case for the root $e^{(2)}-e^{(1)}$. In this case a root vector is given by the matrix unit $E_{3,2}$. So, as in the previous case, we consider the action of the 1-parameter subgroup on $\mathbf{x}\in X_{(2,1,1)}$:

$$z(s)=\mathbf{x}\exp(sE_{3,2})=(\mathbf{x}_1,\mathbf{x}_2+s\mathbf{x}_3,\mathbf{x}_3,\mathbf{x}_4)=\begin{pmatrix}1&0&0&1\\0&x+s&1&-1\end{pmatrix}.$$

Then we see that $z(s)$ is already in the normal form

$$\mathbf{x}(s)=\begin{pmatrix}1&0&0&1\\0&x(s)&1&-1\end{pmatrix},\qquad x(s)=x+s,$$

and hence $F(z(s),\alpha)=F(\mathbf{x}(s),\alpha)$. Now we can compute the left hand side of (2.11) for the root $e^{(2)}-e^{(1)}$

$$\begin{aligned}L_{e^{(2)}-e^{(1)}}F(\mathbf{x},\alpha)&=\frac{d}{ds}F(\mathbf{x}(s),\alpha)|_{s=0}\\&=\frac{\partial F(\mathbf{x},\alpha)}{\partial x}\frac{dx(s)}{ds}|_{s=0}=\frac{d}{dx}F(\mathbf{x},\alpha).\end{aligned}$$

So (2.11) implies

$$\frac{d}{dx}F(\mathbf{x},\alpha)=\alpha_2F(\mathbf{x},\alpha+e^{(2)}-e^{(1)}).$$

Taking into account that

$$\frac{C(\alpha+e^{(2)}-e^{(1)})}{C(\alpha)}=\frac{\alpha_3+1}{\alpha_3+\alpha_4+2}=\frac{a}{c},$$

we get the contiguity relation:

$$\frac{d}{dx}\,{}_1F_1(a,c;x)=\frac{a}{c}\cdot\,{}_1F_1(a+1,c+1;x)$$

with the contiguity operator $L_{e^{(2)}-e^{(1)}}=\frac{d}{dx}$.

3.3 Bessel case

From (2.1) and (2.14), we see that the Bessel integral $I_B(c;x)$ is identified with Gelfand's HGF of type $\lambda=(2,2)$ on the realization $X=X_\lambda\subset Z_\lambda$ of the quotient space $\mathrm{GL}(2)\backslash Z_\lambda/H_\lambda$. We adopt the notation in Subsection 2.3 for numbering the entries of matrices. We have

$$F(\mathbf{x},\alpha)=I_B(c;x),\tag{3.11}$$

where $\alpha=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=(c-1,1,-c-1,1)$ and $\mathbf{x}=(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)=\begin{pmatrix}1&0&0&-x\\0&1&1&0\end{pmatrix}$. Then Proposition 2.6 gives the following result.

Proposition 3.4. *The contiguity relations for Bessel integral $I_B(c,x)$ are given as follows according to the roots $\Delta = \{\pm(e^{(1)} - e^{(2)})\}$:*

$$L_{e^{(1)}-e^{(2)}} I_B(c;x) = I_B(c+1;x),$$
$$L_{e^{(2)}-e^{(1)}} I_B(c;x) = I_B(c-1;x),$$

where

$$L_{e^{(1)}-e^{(2)}} = -\frac{d}{dx}, \quad L_{e^{(2)}-e^{(1)}} = x\frac{d}{dx} + c.$$

Proof. Strategy is the same as in the Kummer's case. To show the first relation, note that the root vector for the root $e^{(1)} - e^{(2)}$ is given by the matrix unit $E_{1,4}$ and consider the action of the 1-parameter subgroup $s \mapsto \exp(sE_{1,4})$ on $\mathbf{x}$:

$$z(s) = \mathbf{x}\exp(sE_{1,4}) = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 + s\mathbf{x}_1) = \begin{pmatrix} 1 & 0 & 0 & -(x-s) \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Since $z(s)$ is already the normal form

$$\mathbf{x}(s) = \begin{pmatrix} 1 & 0 & 0 & -x(s) \\ 0 & 1 & 1 & 0 \end{pmatrix}, \qquad x(s) = x - s,$$

we have $F(z(s),\alpha) = F(\mathbf{x}(s),\alpha)$ and we can compute the left hand side of (2.11) as

$$L_{e^{(1)}-e^{(2)}} F(\mathbf{x},\alpha) = \frac{d}{ds}F(z(s),\alpha)|_{s=0} = \frac{\partial F(\mathbf{x},\alpha)}{\partial x}\frac{dx(s)}{ds}|_{s=0} = -\frac{d}{dx}F(\mathbf{x},\alpha).$$

So (2.11) implies

$$\left\{-\frac{d}{dx}\right\} F(\mathbf{x},\alpha) = \alpha_4 F(\mathbf{x}, \alpha + e^{(1)} - e^{(2)}).$$

Noting that $\alpha = (\alpha_1,\alpha_2,\alpha_3,\alpha_4) = (c-1, 1, -c-1, 1)$ and that $\alpha \mapsto \alpha + e^{(1)} - e^{(2)}$ is the change $\alpha_1 \mapsto \alpha_1 + 1, \alpha_3 \mapsto \alpha_3 - 1$, we have the contiguity relation

$$-\frac{d}{dx} I_B(c;x) = I_B(c+1;x)$$

with the contiguity operator $L_{e^{(1)}-e^{(2)}} = -\frac{d}{dx}$.

Next we prove the second half of the proposition briefly. The root vector for $e^{(2)} - e^{(1)}$ is the matrix unit $E_{3,2}$.The action of the 1-parameter subgroup on $\mathbf{x}$ is

$$z(s) = \mathbf{x}\exp(sE_{3,2}) = (\mathbf{x}_1, \mathbf{x}_2 + s\mathbf{x}_3, \mathbf{x}_3, \mathbf{x}_4) = \begin{pmatrix} 1 & 0 & 0 & -x \\ 0 & 1+s & 1 & 0 \end{pmatrix}.$$

Normalization $\mathbf{x}(s)$ of $z(s)$ is

$$\mathbf{x}(s) = \begin{pmatrix} 1 & 0 & 0 & -x(s) \\ 0 & 1 & 1 & 0 \end{pmatrix}, \qquad x(s) = (1+s)x,$$

where $\mathbf{x}(s) = g^{-1}z(s)h$ with $g = \begin{pmatrix} h_1 & 0 \\ 0 & 1 \end{pmatrix}$, $h = \mathrm{diag}(h_1, h_1, 1, 1) \in H_{(2,2)}$with $h_1 = (1+s)^{-1}$. Then Proposition 2.5 tells us

$$F(z(s), \alpha) = \det(g)^{-1}\chi(h, \alpha)^{-1}F(\mathbf{x}(s), \alpha) = (1+s)^{\alpha_1+1}F(\mathbf{x}(s), \alpha). \tag{3.12}$$

Then using (3.12), the operator $L_{e^{(2)}-e^{(1)}}$ can be computed as

$$\begin{aligned} L_{e^{(2)}-e^{(1)}}F(\mathbf{x}, \alpha) &= \frac{d}{ds}(1+s)^{\alpha_1+1}F(\mathbf{x}(s), \alpha)|_{s=0} \\ &= \left\{ x\frac{d}{dx} + (\alpha_1 + 1) \right\} F(\mathbf{x}, \alpha). \end{aligned}$$

So (2.11) reads as

$$\left\{ x\frac{d}{dx} + (\alpha_1 + 1) \right\} F(\mathbf{x}, \alpha) = \alpha_2 F(\mathbf{x}, \alpha + e^{(2)} - e^{(1)}),$$

which gives the contiguity relation

$$\left\{ x\frac{d}{dx} + c \right\} I_B(c; x) = I_B(c-1; x)$$

with the contiguity operator $L_{e^{(2)}-e^{(1)}} = x\frac{d}{dx} + c$.

3.4 Hermite-Weber case

By comparing (2.1) and (2.15), Hermite-Weber integral $I_{HW}(a; x)$ is identified with Gelfand's HGF of type $\lambda = (3, 1)$ on the realization $X = X_\lambda \subset Z_\lambda$ of the quotient space $\mathrm{GL}(2)\backslash Z_\lambda/H_\lambda$. We use the similar notation as above for numbering the entries of matrices. Note that an element $\mathbf{x} \in X_{(3,1)}$ is of the form $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \begin{pmatrix} 1 & 0 & 0 & -x \\ 0 & 1 & 1 & 0 \end{pmatrix}$ and

$$F(\mathbf{x}, \alpha) = I_{HW}(a; x), \tag{3.13}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (a-1, 0, 1, -a-1)$. Then Proposition 2.6 gives the following result.

Proposition 3.5. *The contiguity relations for the Hermite-Weber integral $I_{HW}(a; x)$ are given as follows according to the roots $\Delta = \{\pm(e^{(1)} - e^{(2)})\}$:*

$$\begin{aligned} L_{e^{(1)}-e^{(2)}}I_{HW}(a; x) &= (-a-1)I_{HW}(a+1; x), \\ L_{e^{(2)}-e^{(1)}}I_{HW}(a; x) &= I_{HW}(a-1; x), \end{aligned}$$

where

$$L_{e^{(1)}-e^{(2)}} = \frac{d}{dx} - x, \quad L_{e^{(2)}-e^{(1)}} = \frac{d}{dx}.$$

Proof. We proceed in the same way as in the Bessel case. Note that the matrix units $E_{1,4}$ and $E_{4,3}$ are root vectors for the roots $e^{(1)} - e^{(2)}$ and $e^{(2)} - e^{(1)}$, respectively.

1) $e^{(1)} - e^{(2)}$ case. Consider the action of 1-parameter subgroup $s \mapsto \exp(sE_{1,4})$ on $\mathbf{x} \in X$:

$$z(s) = \mathbf{x}\exp(sE_{1,4}) = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 + s\mathbf{x}_1) = \begin{pmatrix} 1 & 0 & 0 & s \\ 0 & 1 & x & 1 \end{pmatrix}.$$

We check that $z(s)$ is normalized to

$$\mathbf{x}(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & x(s) & 1 \end{pmatrix}, \qquad x(s) = x + s \tag{3.14}$$

by the action of $\mathrm{GL}(2) \times H_{(3,1)}$. Take

$$g = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & h_2 & h_3 & \\ & 1 & h_2 & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

and consider $g^{-1}z(s)h$. Then we have

$$g^{-1}z(s)h = \begin{pmatrix} 1 & h_2 - s & h_3 - sh_2 - sx & 0 \\ 0 & 1 & h_2 + x & 1 \end{pmatrix}.$$

Taking h with $h_2 = s, h_3 = s^2 + sx$, we see that $g^{-1}z(s)h$ is of the normal form given in (3.14). It follows from Proposition 2.5 that

$$\begin{aligned} F(z(s), \alpha) &= \det(g)^{-1}\chi(h, \alpha)^{-1}F(\mathbf{x}(s), \alpha) \\ &= \exp(-sx - \frac{1}{2}s^2)F(\mathbf{x}(s), \alpha). \end{aligned} \tag{3.15}$$

Then by using (3.15), the left hand side of (2.11) in this case is computed as

$$\begin{aligned} L_{e^{(1)}-e^{(2)}}F(\mathbf{x}, \alpha) &= \frac{d}{ds}F(z(s), \alpha)|_{s=0} \\ &= \frac{d}{ds}\exp(-sx - \frac{1}{2}s^2)|_{s=0}F(\mathbf{x}, \alpha) + \frac{\partial F(\mathbf{x}, \alpha)}{\partial x}\frac{dx(s)}{ds}|_{s=0} \\ &= \left\{\frac{d}{dx} - x\right\}F(\mathbf{x}, \alpha). \end{aligned}$$

So (2.11) implies

$$\left\{\frac{d}{dx} - x\right\}F(\mathbf{x}, \alpha) = \alpha_4 F(\mathbf{x}, \alpha + e^{(1)} - e^{(2)}).$$

Noting that $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (a - 1, 0, 1, -a - 1)$, we have the contiguity relation

$$\left\{\frac{d}{dx} - x\right\}I_{HW}(a; x) = (-a - 1)I_{HW}(a + 1; x).$$

2) $e^{(2)}-e^{(1)}$ case. Consider the action of the 1-parameter subgroup $\{\exp(sE_{4,3})\}$ on $\mathbf{x} \in X$:

$$z(s) = \mathbf{x}\exp(sE_{4,3}) = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 + s\mathbf{x}_4, \mathbf{x}_4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & x+s & 1 \end{pmatrix}.$$

Since $z(s)$ is already a normal form

$$\mathbf{x}(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & x(s) & 1 \end{pmatrix}, \qquad x(s) = x + s.$$

From Proposition 2.5 we have $F(z(s), \alpha) = F(\mathbf{x}(s), \alpha)$ and the left hand side of (2.11) in this case is computed as

$$L_{e^{(2)}-e^{(1)}} F(\mathbf{x}, \alpha) = \frac{d}{ds} F(z(s), \alpha)|_{s=0} = \frac{\partial}{\partial x} F(\mathbf{x}, \alpha) \frac{dx(s)}{ds}|_{s=0} = \frac{d}{dx} F(\mathbf{x}, \alpha).$$

So (2.11) implies

$$\frac{d}{dx} F(\mathbf{x}, \alpha) = \alpha_3 F(\mathbf{x}, \alpha + e^{(2)} - e^{(1)}).$$

Noting that $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (a-1, 0, 1, -a-1)$ and that $\alpha \mapsto \alpha + e^{(2)} - e^{(1)}$ implies $\alpha_1 \mapsto \alpha_1 - 1, \alpha_4 \mapsto \alpha_4 + 1$, we have the contiguity relation

$$\frac{d}{dx} I_{HW}(a; x) = I_{HW}(a-1; x).$$

4 Contiguity of beta and gamma functions

In this section, we show that the famous contiguity relation (recurrence relation) $\Gamma(a+1) = a\Gamma(a)$ for the gamma function can be derived from that for Gelfand's HGF. We also establish the similar assertion for the beta function.

4.1 Beta and Gamma as Gelfand's HGF

The beta function and the gamma function are defined by the integrals

$$B(a, b) = \int_0^1 u^{a-1}(1-u)^{b-1} du, \tag{4.1}$$

$$\Gamma(a) = \int_0^\infty u^{a-1} e^{-u} du, \tag{4.2}$$

which converge for $\mathrm{Re}(a), \mathrm{Re}(b) > 0$ and for $\mathrm{Re}(a) > 0$, respectively, and define holomorphic functions there. We explain these functions can also be identified with Gelfand's HGF on the Grassmannian $\mathrm{Gr}(2, 3)$ corresponding respectively to the partitions $\lambda = (1, 1, 1)$ and $\lambda = (2, 1)$. To the partition λ, we associate the group H_λ as

$$H_{(1,1,1)} = \left\{ \begin{pmatrix} h_1 & & \\ & h_2 & \\ & & h_3 \end{pmatrix} \mid h_1, h_2, h_3 \neq 0 \right\},$$
$$H_{(2,1)} = \left\{ \begin{pmatrix} h_1 & h_2 & \\ & h_1 & \\ & & h_3 \end{pmatrix} \mid h_1, h_3 \neq 0 \right\}.$$

Then the characters $\chi_\lambda : \tilde{H}_\lambda \to \mathbb{C}^\times$ we use are given by

$$\chi_{(1,1,1)}(h) = h_1^{\alpha_1} h_2^{\alpha_2} h_3^{\alpha_3}, \qquad \alpha_1 + \alpha_2 + \alpha_3 = -2,$$
$$\chi_{(2,1)}(h) = h_1^{\alpha_1} \exp\left(\alpha_2 \frac{h_2}{h_1}\right) h_3^{\alpha_3}, \qquad \alpha_1 + \alpha_3 = -2, \alpha_2 = -1.$$

Then the spaces on which Gelfand's HGF are defined for $\lambda = (1,1,1), (2,1)$ are

$$Z_{(1,1,1)} = \{(z_1, z_2, z_3) \in \mathrm{Mat}(2,3) \mid \det(z_i, z_j) \neq 0 \ (i \neq j)\},$$
$$Z_{(2,1)} = \{(z_1, z_2, z_3) \in \mathrm{Mat}(2,3) \mid \det(z_1, z_2) \neq 0, \ \det(z_1, z_3) \neq 0\}.$$

The space Z_λ is invariant by the action of $\mathrm{GL}(2)$ and H_λ defined by the left and right multiplication of matrices and we can consider the quotient spaces $\mathrm{GL}(2)\backslash Z_\lambda / H_\lambda$. The quotient space consists of one point and are realized as a subset X_λ of Z_λ as

$$X_{(1,1,1)} = \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \right\} \subset Z_{(1,1,1)},$$
$$X_{(2,1)} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \right\} \subset Z_{(2,1)}.$$

The restriction of Gelfand's HGF for $\lambda = (1,1,1)$ on $X_{(1,1,1)} \ni \mathbf{x} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ gives the beta function

$$\begin{aligned} F(\mathbf{x}, \alpha) &= \int_C (\vec{u}\mathbf{x}_1)^{\alpha_1} (\vec{u}\mathbf{x}_2)^{\alpha_2} (\vec{u}\mathbf{x}_3)^{\alpha_3} du \\ &= \int_C 1^{\alpha_1} u^{\alpha_2} (1-u)^{\alpha_3} du = B(\alpha_2 + 1, \alpha_3 + 1), \end{aligned} \tag{4.3}$$

where we take $C = \overrightarrow{0,1}$. Similarly, the restriction of Gelfand's HGF of type $\lambda = (2,1)$ on $X_{(2,1)} \ni \mathbf{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ gives the gamma function:

$$\begin{aligned} F(\mathbf{x}, \alpha) &= \int_C (\vec{u}\mathbf{x}_1)^{\alpha_1} \exp\left(\alpha_2 (\vec{u}\mathbf{x}_2)/(\vec{u}\mathbf{x}_1)\right) (\vec{u}\mathbf{x}_3)^{\alpha_3} du \\ &= \int_C 1^{\alpha_1} e^{\alpha_2 u} u^{\alpha_3} du = \Gamma(\alpha_3 + 1), \end{aligned} \tag{4.4}$$

where we take $C = \overrightarrow{0, \infty}$.

4.2 Beta case

Note that the correspondence between the weights of χ and the independent variables a, b of the beta is

$$\alpha_1 = -a - b, \quad \alpha_2 = a - 1, \quad \alpha_3 = b - 1.$$

The contiguity relation is linked with the action of the 1-parameter subgroup generated by a root vector in the root space decomposition of Lie algebra $\mathfrak{gl}(3)$:

$$\mathfrak{gl}(3) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

with the set of roots $\Delta = \{e^{(i)} - e^{(j)} \mid 1 \le i \ne j \le 3\}$ and the root space $\mathfrak{g}_{e^{(i)}-e^{(j)}} = \mathbb{C} \cdot E_{i,j}$, where $E_{i,j}$ is the (i,j) matrix unit. Then Proposition 2.6 tells us that we can obtain the contiguity relation

$$L_{e^{(i)}-e^{(j)}} F(\mathbf{x}, \alpha) = \alpha_j F(\mathbf{x}, \alpha + e^{(i)} - e^{(j)}) \tag{4.5}$$

from the action of the 1-parameter subgroup generated by the root vector $E_{i,j}$. Here, as explained earlier, the operator $L_{e^{(i)}-e^{(j)}}$ is defined by

$$(L_{e^{(i)}-e^{(j)}} \cdot f)(z) = \frac{d}{ds} f(z \exp(sE_{i,j}))|_{s=0} \tag{4.6}$$

and $\alpha \mapsto \alpha + e^{(i)} - e^{(j)}$ implies that α_i, α_j are changed as $\alpha_i \mapsto \alpha_i + 1, \alpha_j \mapsto \alpha_j - 1$ and the rest is left unchanged.

Proposition 4.1. *The operator $L_{e^{(i)}-e^{(j)}}$ acts on $F(\mathbf{x}, \alpha)$ as a multiplication by the constant $A_{i,j}$ and the relation (4.5) is written as*

$$A_{i,j} F(\mathbf{x}, \alpha) = \alpha_j F(\mathbf{x}, \alpha + e^{(i)} - e^{(j)}),$$

which is rewritten into the contiguity relation for the beta function as described in the table. In particular, it gives the relation

$$B(a+1, b) = \frac{a}{a+b} B(a, b), \quad B(a, b+1) = \frac{b}{a+b} B(a, b).$$

Roots	$A_{i,j}$	a	b	Contiguity relation
$e^{(1)} - e^{(2)}$	$-(\alpha_1 + 1)$	-1		$B(a,b) = \frac{a-1}{a+b-1} B(a-1, b)$
$e^{(2)} - e^{(1)}$	$-(\alpha_2 + 1)$	$+1$		$B(a,b) = \frac{a+b}{a} B(a+1, b)$
$e^{(1)} - e^{(3)}$	$-(\alpha_1 + 1)$		-1	$B(a,b) = \frac{b-1}{a+b-1} B(a, b-1)$
$e^{(3)} - e^{(1)}$	$-(\alpha_3 + 1)$		$+1$	$B(a,b) = \frac{a+b}{b} B(a, b+1)$
$e^{(2)} - e^{(3)}$	$\alpha_2 + 1$	$+1$	-1	$B(a,b) = \frac{b-1}{a} B(a+1, b-1)$
$e^{(3)} - e^{(2)}$	$\alpha_3 + 1$	-1	$+1$	$B(a,b) = \frac{a-1}{b} B(a-1, b+1)$

Proof. We discuss the cases of the roots $\pm(e^{(1)} - e^{(2)})$. Consider the 1-parameter subgroup generated by the root vector $E_{1,2}$ for the root $e^{(1)} - e^{(2)}$ and its action on $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$:

$$\begin{aligned} z(s) = \mathbf{x} \exp(sE_{1,2}) &= (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \begin{pmatrix} 1 & s & \\ & 1 & \\ & & 1 \end{pmatrix} \\ &= (\mathbf{x}_1, \mathbf{x}_2 + s\mathbf{x}_1, \mathbf{x}_3) = \begin{pmatrix} 1 & s & 1 \\ 0 & 1 & -1 \end{pmatrix}. \end{aligned}$$

Next we normalize $z(s)$ to the normal form $\mathbf{x}$. Put $g_1 = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ and $h = \operatorname{diag}(1, h_2, h_3)$ and consider

$$g_1^{-1} z(s) h = \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & s & 1 \\ 0 & 1 & -1 \end{pmatrix} h = \begin{pmatrix} 1 & 0 & (1+s)h_3 \\ 0 & h_2 & -h_3 \end{pmatrix}.$$

Further we put $g_2 = \operatorname{diag}(1, h_2)$ and consider

$$g_2^{-1} g_1^{-1} z(s) h = \begin{pmatrix} 1 & 0 & (1+s)h_3 \\ 0 & 1 & -h_2^{-1} h_3 \end{pmatrix}.$$

Taking h with $h_2 = h_3 = (1+s)^{-1}$, we get

$$g_2^{-1} g_1^{-1} z(s) h = \mathbf{x}.$$

It follows from Proposition 2.5 that

$$F(z(s), \alpha) = \det(g_1 g_2)^{-1} \chi(h, \alpha)^{-1} F(\mathbf{x}, \alpha) = (1+s)^{\alpha_2 + \alpha_3 + 1} F(\mathbf{x}, \alpha). \tag{4.7}$$

Since the left hand side of the contiguity relation (4.5) is given by the operator (4.6) with $(i, j) = (1, 2)$, we compute it using (4.7) as

$$\begin{aligned} L_{e^{(1)} - e^{(2)}} F(\mathbf{x}, \alpha) &= \frac{d}{ds} F(z(s), \alpha)|_{s=0} \\ &= \frac{d}{ds} (1+s)^{\alpha_2 + \alpha_3 + 1}|_{s=0} \cdot F(\mathbf{x}, \alpha) \\ &= (\alpha_2 + \alpha_3 + 1) F(z, \alpha). \end{aligned}$$

This implies that $A_{1,2} = \alpha_2 + \alpha_3 + 1 (= -\alpha_1 - 1)$ and (4.5) reads

$$F(\mathbf{x}, \alpha) = \frac{\alpha_2}{\alpha_2 + \alpha_3 + 1} F(\mathbf{x}, \alpha + e^{(1)} - e^{(2)}).$$

Taking into account that $\alpha_1 = -a - b, \alpha_2 = a - 1, \alpha_3 = b - 1$, this identity is translated to the contiguity

$$B(a, b) = \frac{a-1}{a+b-1} B(a-1, b).$$

The case for the root $-(e^{(1)}-e^{(2)})$ is treated in a similar way. The action of the 1-parameter subgroup $\exp(sE_{2,1})$ on $\mathbf{x}$ is

$$z(s)=\mathbf{x}\exp(sE_{2,1})=\begin{pmatrix}1&0&1\\ s&1&-1\end{pmatrix}.$$

Normalization of $z(s)$ to the normal form is $g_2^{-1}g_1^{-1}z(s)h=\mathbf{x}$ is with

$$g_1=\begin{pmatrix}1&\\ s&1\end{pmatrix},\quad g_2=\begin{pmatrix}h_1&\\ &1\end{pmatrix},\quad h=\begin{pmatrix}h_1&&\\ &1&\\ &&h_3\end{pmatrix}$$

and $h_1=h_3=(1+s)^{-1}$. It follows from Proposition 2.5 that

$$F(z(s),\alpha)=\det(g_1g_2)^{-1}\chi(h,\alpha)^{-1}F(\mathbf{x},\alpha)=(1+s)^{\alpha_1+\alpha_3+1}F(\mathbf{x},\alpha). \tag{4.8}$$

Then

$$L_{e^{(2)}-e^{(1)}}F(\mathbf{x},\alpha)=\frac{d}{ds}(1+s)^{\alpha_1+\alpha_3+1}|_{s=0}\cdot F(z,\alpha)=(\alpha_1+\alpha_3+1)F(z,\alpha).$$

So $A_{2,1}=\alpha_1+\alpha_3+1=-\alpha_2-1$ and (4.5) implies

$$F(\mathbf{x},\alpha)=\frac{\alpha_1}{\alpha_1+\alpha_3+1}F(z,\alpha+e^{(2)}-e^{(1)}).$$

Noting $(\alpha_1,\alpha_2,\alpha_3)=(-a-b,a-1,b-1)$, this identity is translated to the contiguity

$$B(a,b)=\frac{a+b}{a}B(a+1,b).$$

4.3 Gamma case

We establish a result for the gamma function similar to Proposition 4.1. Note that the correspondence between the weights of χ and the independent variable a of the gamma function is

$$\alpha_1=-a-1,\quad \alpha_2=-1,\quad \alpha_3=a-1.$$

The contiguity relation is linked with the action of 1-parameter subgroup generated by a root vector in the generalized root space decomposition of Lie algebra $\mathfrak{gl}(3)$:

$$\mathfrak{gl}(3)=\mathfrak{h}\oplus\mathfrak{g}_{e^{(1)}-e^{(2)}}\oplus\mathfrak{g}_{e^{(2)}-e^{(1)}}$$

with the set of roots $\Delta=\{\pm(e^{(1)}-e^{(2)})\}$. The root vectors corresponding to the roots $e^{(1)}-e^{(2)}$ and $e^{(2)}-e^{(1)}$ are complex constant multiples of $E_{e^{(1)}-e^{(2)}}:=E_{1,3}$ and $E_{e^{(2)}-e^{(1)}}:=E_{3,2}$, respectively. Then Proposition 2.6 tells us that the contiguity relation for $F(\mathbf{x},\alpha)=\Gamma(\alpha_3+1)$ has the form

$$L_{e^{(i)}-e^{(j)}}F(\mathbf{x},\alpha)=\beta_j\cdot F(\mathbf{x},\alpha+e^{(i)}-e^{(j)}), \tag{4.9}$$

where $\beta_1:=\alpha_2,\beta_2:=\alpha_3$ and the operator $L_{e^{(i)}-e^{(j)}}$ is defined by

$$(L_{e^{(i)}-e^{(j)}}\cdot f)(z)=\frac{d}{ds}f(z\exp(sE_{e^{(i)}-e^{(j)}}))|_{s=0}. \tag{4.10}$$

Proposition 4.2. *The operator $L_{e^{(i)}-e^{(j)}}$ acts on $F(\mathbf{x},\alpha)$ as a multiplication by the constant $A_{i,j}$ and the relation (4.9) is written as*

$$A_{i,j}F(\mathbf{x},\alpha)=\beta_j\cdot F(\mathbf{x},\alpha+e^{(i)}-e^{(j)}),$$

which is rewritten as the contiguity relation for the gamma function described in the following table. In particular, it gives the relation $\Gamma(a+1)=a\Gamma(a)$.

Roots	$A_{i,j}$	a	Contiguity relation
$e^{(1)}-e^{(2)}$	1	-1	$\Gamma(a)=(a-1)\Gamma(a-1)$
$e^{(2)}-e^{(1)}$	$-(\alpha_3+1)$	$+1$	$-a\Gamma(a)=-\Gamma(a+1)$

Proof. The strategy of proof is the same as in the beta case. Consider the 1-parameter subgroup generated by the root vector $E_{1,3}$ for the root $e^{(1)}-e^{(2)}$ and its action on $\mathbf{x}=(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)=\begin{pmatrix}1&0&0\\0&1&1\end{pmatrix}$ and obtain

$$z(s)=\mathbf{x}\exp(sE_{1,3})=(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3+s\mathbf{x}_1)=\begin{pmatrix}1&0&s\\0&1&1\end{pmatrix}.$$

We normalize $z(s)$ to the normal form $\mathbf{x}$. Put

$$g=\begin{pmatrix}1&s\\0&1\end{pmatrix},\quad h=\begin{pmatrix}1&h_2&\\&1&\\&&1\end{pmatrix}$$

and consider

$$g^{-1}z(s)h=\begin{pmatrix}1&h_2-s&0\\0&1&1\end{pmatrix}.$$

Then we have $g^{-1}z(s)h=\mathbf{x}$ by taking h with $h_2=s$. It follows from Proposition 2.6 that

$$F(z(s),\alpha)=\det(g)^{-1}\chi(h,\alpha)^{-1}F(\mathbf{x},\alpha)=e^sF(\mathbf{x},\alpha). \tag{4.11}$$

Since the contiguity relation is given by (4.9), (4.10), we compute the left hand side of (4.9) using (4.11) as

$$L_{e^{(1)}-e^{(2)}}F(\mathbf{x},\alpha)=\frac{d}{ds}F(z(s),\alpha)|_{s=0}=\frac{d}{ds}e^s|_{s=0}\cdot F(\mathbf{x},\alpha)=F(\mathbf{x},\alpha).$$

So $L_{e^{(1)}-e^{(2)}}$ acts on $F(\mathbf{x},\alpha)$ as a multiplication by the constant $A_{1,2}=1$ and (4.9) implies

$$F(\mathbf{x},\alpha)=\beta_2F(z,\alpha+e^{(1)}-e^{(2)}).$$

Taking into account the fact that $(\alpha_1,\alpha_2,\alpha_3)=(-a-1,-1,a-1)$, $\alpha+e^{(1)}-e^{(2)}=(\alpha_1+1,\alpha_2,\alpha_3-1)$ and $\beta_2=\alpha_3$, this identity reads as the recurrence relation

$$\Gamma(a)=(a-1)\Gamma(a-1).$$

The case for the root $e^{(2)}-e^{(1)}$ is shown in the same way, so we omit it.

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